# Chapter 9

# ROTATIONAL MOTION -- PART I I

## A.) Torque:

**1.)** So far, we have developed rotational counterparts for displacement, velocity, acceleration, and mass. It is now time to consider the rotational counterpart to force.

When a net force is applied to an object, the object accelerates (Newton's Second Law). *Torque* is the rotational counterpart to *force* in the sense that when a net torque is applied to a body, the body *angularly* accelerates. While force is applied *in a particular direction*, torque is applied *about a point* (the point of interest is usually on the body's axis of rotation). Torque calculations were briefly discussed in Chapter 1 (the idea of a torque was used there as an example of a vector *cross product* operation). We will go into more depth here.

**2.)** The easiest way to understand the concept of a torque is with an example.

**a.)** A force ***F*** is applied to a wrench a distance ***r*** from the axis of rotation (see Figure 9.1). From experience, it should be obvious that:

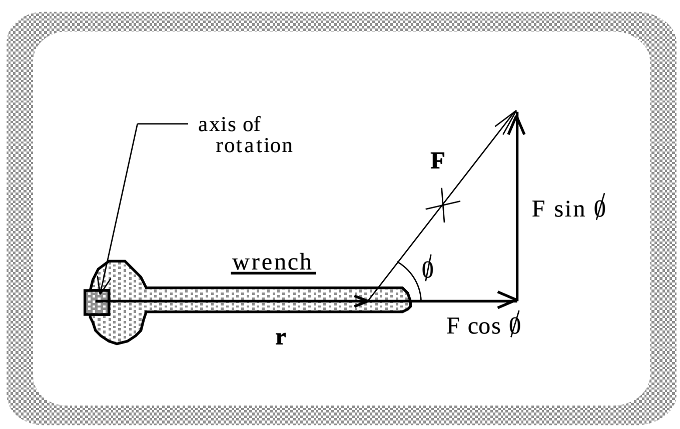


FIGURE 9.1

**i.)** The greater $|\mathbf{r}|$ is, the less difficult it is to *angularly accelerate* the bolt;

**ii.)** The greater $|\mathbf{F}|$ is, the less difficult it is to *angularly accelerate* the bolt; and

**iii.)** The *force component* that makes the bolt angularly accelerate is the component *perpendicular* to the line of ***r*** (i.e., $|\mathbf{F}| \sin \phi$).

**b.)** As *ease of rotation* is related to $|\boldsymbol{r}|$ and $|\mathbf{F}| sin\ \phi$, the product of those two variables is deemed important enough to be given a special name--*torque* ($\boldsymbol{\Gamma}$). In short, the *magnitude* of the *torque* applied by $\boldsymbol{F}$ about some point will be $|\boldsymbol{\Gamma}| = |\boldsymbol{r} x \boldsymbol{F}|$. As a *vector*, torque is defined as:

$$\mathbf{\Gamma} = \mathbf{r} \text{ x } \mathbf{F}.$$

**Note:** It is not unusual to find physics texts making statements like, "a force $\boldsymbol{F}$ applies a torque about the axis of rotation." This can be confusing because, by definition, torques are not applied about axes--they are applied about points. A more accurate way to make the statement would be to say, "a force $\boldsymbol{F}$ applies a torque about a point that is both in the plane of the paper and on the axis of rotation." Unfortunately, although this is technically correct, it is also wordy and cumbersome. As a consequence, physicists shorten such statements to, "a force $\boldsymbol{F}$ applies a torque about the axis of rotation."

There is nothing wrong with this shorthand description as long as you understand the assumption being made when torque calculations are termed this way.

Bottom line: From here on out, you will be expected to know what "take the torque about the axis of rotation" means.

**3.)** In the first chapter we found that a *cross product* is a vector manipulation involving two vectors (say $\boldsymbol{r}$ and $\boldsymbol{F}$). It generates a third vector whose *magnitude* is numerically equal to the product of:

**a.)** The *magnitude* of one of the vectors (say $|\boldsymbol{r}|$ in this case), and

**b.)** The *magnitude* of the second-vector's-component that runs *perpendicular* to the first vector (in this case, $|\boldsymbol{F}| sin\ \phi$).

**c.)** Assuming the vector information is in polar notation, the *magnitude* of the torque calculation will be the *magnitude* of a *cross product*, or:

$$\begin{aligned} |\mathbf{\Gamma}_F| &= |\mathbf{r} \text{x} \mathbf{F}| \\ &= |\mathbf{r}|\, |\mathbf{F}| \sin\phi, \end{aligned}$$

where $\phi$ is the angle between the *line of* $\boldsymbol{r}$ and the *line of* $\boldsymbol{F}$.

**4.)** The *direction* of the *cross product* is *perpendicular* to the plane defined by the two vectors. In the case of a torque produced by an ***r*** and ***F*** vector in the *x-y plane*, this direction is along the *z axis* in the ***k*** *direction*. That is fortunate. Remembering that the direction of an *angular velocity* and *angular acceleration* vector is along the axis of rotation, a torque that makes an object rotate in the *x-y plane* should have a direction perpendicular to the *x-y plane* (i.e., in the direction of the axis of rotation about which the angular acceleration is taking place). That is exactly the direction the *cross product* gives us.

**5.)** There are three ways to calculate a *cross product* and, hence, a torque. All three will be presented below in the context of the following problem: A 10 newton force is applied at a $60^o$ angle to a giant wrench 3 meters from the axis of rotation (see Figure 9.2). How much torque does the force apply about a point on the central axis of the bolt (i.e., on the axis of rotation) in the plane of the wrench?

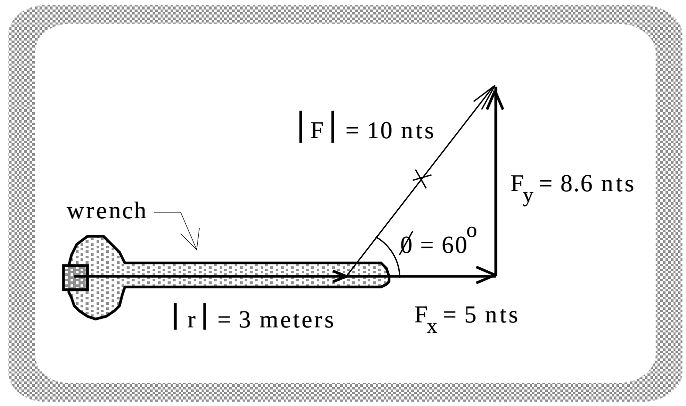


FIGURE 9.2

**a.)** The "definition" approach: Take the definition of a cross product and apply it to the situation. Doing so yields:

$$\begin{aligned} |\mathbf{\Gamma}_F| &= |\mathbf{r} \mathrm{x} \mathbf{F}| \\ &= |\mathbf{r}| \quad |\mathbf{F}| \quad \sin\phi, \\ &= (3\text{ m})\ (10\text{ nts})\sin 60^o \\ &= 25.98\text{ nt}\cdot\text{m}. \end{aligned}$$

The direction is determined using the right-hand rule. Doing so yields a +***k*** *direction*. As our rotation is one-dimensional (i.e., there is only one axis about which the rotation occurs) in the *x-y plane*, we don't need to include the ***k*** *unit vector*. We do need to include the "+" sign (it tells us that the torque will attempt to *angularly accelerate* the object in the *counterclockwise* direction). The end result is, therefore:

$$\Gamma_F = +25.98\text{ nt}\cdot\text{m}.$$

**b.)** The "$r_\perp$" approach: We know that the *magnitude* of a cross product is equal to the *magnitude of one vector* times the *perpendicular* *component of the second vector* (i.e., the component of the second vector perpendicular to the line of the first vector). If we let ***F*** be the first vector, the "perpendicular component of the second vector" will be the component of ***r*** perpendicular to the *line of **F***. Calling this term $r_\perp$, we have:

$$|\boldsymbol{\Gamma}_F| = (r_\perp)|\mathbf{F}|.$$

**Note 1:** This approach is so commonly used that most texts give $r_\perp$ a special name. They call it *the moment arm*. Using that term, we write, "the *torque* about *Point P* is equal to the *force* times the *moment arm* about *Point P*."

Why is the $r_\perp$ *approach* used so often? Read Note 2!

**Note 2:** Physically, $r_\perp$ is the *shortest distance* between the *point about which the torque is being taken* (usually on the axis of rotation) and the *line of the force*. As it is often easy to determine the *shortest distance* between a *point* and a *line*, this method of calculating torques is very popular.

**Note 3:** Having extolled the virtues of the $r_\perp$ *approach*, it should be pointed out that in this particular problem, the easiest way to determine the torque is by using either the *definition approach* or the approach that will be presented last. Be that as it may, $r_\perp$ is what we are concerned with here!

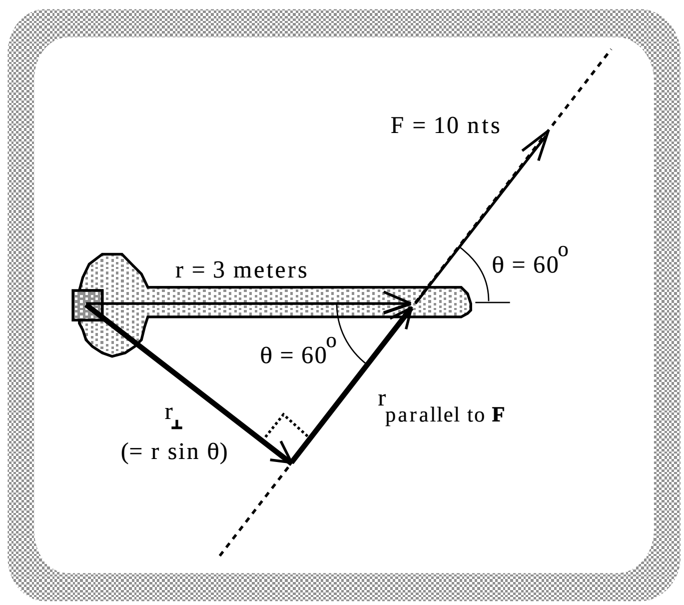


FIGURE 9.3

**CONTINUING:** Consider Figure 9.3. The line of ***F*** has been extended in both directions, allowing us to see the shortest distance between "the *axis of rotation* and the *line-of-the-force*" (i.e., $r_\perp$). With that and a little geometry, we find that:

$$\begin{aligned}\left|\mathbf{\Gamma}_F\right| &= \left|\mathbf{r}\mathrm{x}\mathbf{F}\right| \\ &= \left(r_\perp\right) \quad \left|\mathbf{F}\right| \\ &= [(3\text{ m})(\sin 60^o)]\ (10\text{ nts}) \\ &= 25.98\text{ nt}\cdot\text{m}.\end{aligned}$$

**Note:** The *direction* is determined using either the right-hand rule or your knowledge about *clockwise* versus *counterclockwise* rotations. The final solution is +25.98 nt·m.

**c.)** <u>The "$F_\perp$" approach</u>: We know that the *magnitude* of a cross product is equal to the *magnitude of one vector* times the *perpendicular component of the second vector* (i.e., the component of the second vector perpendicular to the line of the first vector). If we let ***r*** be the first vector, the "perpendicular component of the second vector" will be the component of ***F*** that is perpendicular to the *line of **r***. Calling this component $F_\perp$, we have:

$$\left|\mathbf{\Gamma}_F\right| = \left(F_\perp\right)\left|\mathbf{r}\right|.$$

**Note 1:** This is the flip-side of the $r_\perp$ *approach* and it works in approximately the same way. Extend the line of ***r*** until you can see the component of ***F*** perpendicular to that line. With that information, you simply multiply the magnitude of ***r*** by $F_\perp$.

**Note 2:** This approach is most useful whenever you are *given* the force component perpendicular-to-the-line-of-**r**. Our problem is a good example of such a situation. The vector ***r*** is in the *x direction*. We know $F_\perp$ because it was given to us (look back at Figure 9.2). With that unit vector information, the $F_\perp$ *approach* falls out nicely.

<u>Bottom line</u>: If you are given information in *unit vector notation*, think $F_\perp$ *approach*. It won't always work, but when it does it will work easily.

<u>Continuing</u>: As $F_\perp$ is $F_y$, we can write:

$$\begin{aligned}\left|\mathbf{\Gamma}_F\right| &= \left|\mathbf{r}\mathrm{x}\mathbf{F}\right| \\ &= \left(F_\perp\right) \quad \left|\mathbf{r}\right| \\ &= [8.6\text{ nts}]\ (3\text{ m}) \\ &= 25.8\text{ nt}\cdot\text{m}.\end{aligned}$$

**Note:** The direction is determined using either the right-hand rule or your knowledge about *clockwise* versus *counterclockwise* rotations. The final solution is +25.98 nt·m.

**d.)** Even though the $r_\perp$ *approach* is often used, there is really no one approach that is better than any other. For some problems, the $r_\perp$ approach is a horror. KNOW THEM ALL. It's better to have a choice than to get hung with a problem that doesn't seem to easily work out using the only approach you have learned!

## B.) Rigid Body Equilibrium Problems:

**1.)** There are two kinds of equilibrium: dynamic equilibrium in which a body is moving but not accelerating, and static equilibrium in which a body is at rest and not accelerating. The common denominator is *no acceleration*.

Put another way, if one has equilibrium:

**a.)** The *sum of the forces* acting in the *x direction* must add to zero (if that weren't the case, we would see *translational* acceleration in the *x-direction*);

**b.)** The *sum of the forces* in the *y direction* must add to zero; and

**c.)** The *sum of the torques* acting about any point must add to zero (if that weren't the case, we would see *angular* acceleration).

**2.)** Example: A ladder of length $L$ is positioned against a wall. The wall is *frictionless* and the floor is *frictional*. A man of mass $m_m$ stands on the ladder a distance $L/3$ from the top. If the ladder meets the floor at an angle $\theta$ with the horizontal, and if the ladder's mass is $m_L$, determine the forces acting at the floor and the wall. See Figure 9.4.

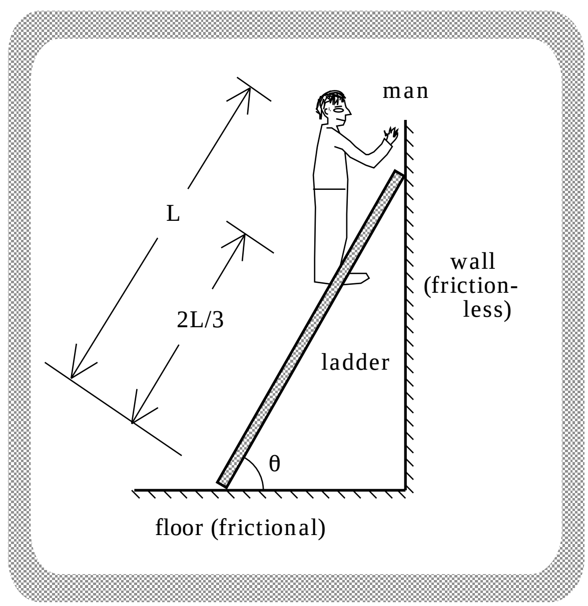

FIGURE 9.4

**a.)** Preliminary Comment #1: In looking at the ladder's contact with the floor, it should be obvious that there is both a *normal* and a *frictional* force acting at that point. Assume, for the moment, that that fact is *not* obvious.

In that case, all we know is that the floor must be providing a *net force* $F_{floor}$ acting at some unknown angle $\phi$ (note that $\phi$ is not $\theta$ here). From experience, we know that unknown forces are easy to deal with, but the math can get dicey when unknown angles are injected into a situation (angles are usually attached to *sine* and *cosine* functions which can make solving simultaneous equations difficult). It would be nice if we could deal with a *force-at-the-floor* problem without having to deal with the unknown angle.

That can be cleverly done by noticing that the *force-at-the-floor* must have $x$ and $y$ *components*. Calling the *horizontal component* $H$ and the *vertical component* $V$, we can solve for those variables. We will still have two unknowns ($H$ and $V$ versus $F_{floor}$ and $\phi$) but we will have traded off for a tidier problem.

**Note:** Another example of a situation in which this ploy will be useful: Consider a lab comprised of a *beam* pinned so as to rotate about an axis at its end (see Figure 9.5). You know *absolutely nothing* about the magnitude and direction of the force acting on the beam at the pin, but you are asked to theoretically determine what that force and angle should be under the circumstances embodied within the set-up. This is a prime example of a situation in which solving for the *components* is preferable to hassling with the actual force vector and its angle.

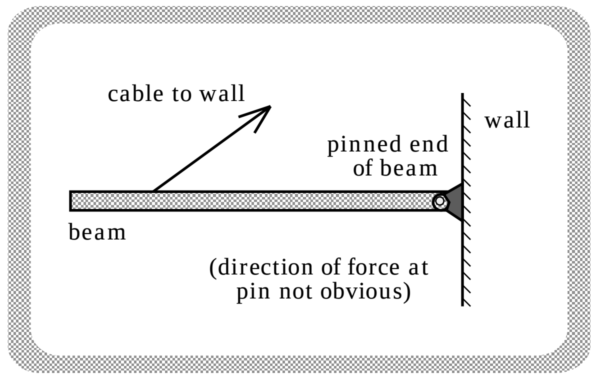


FIGURE 9.5

**b.)** Preliminary Comment #2: Because the wall is frictionless, the force acting at the wall is strictly a *normal force*. As such, we will call that force $N$. Note also that the ladder's weight $m_L g$ is concentrated at the ladder's *center of mass* at $L/2$. This is all shown in the *free body diagram* presented in Figure 9.6.

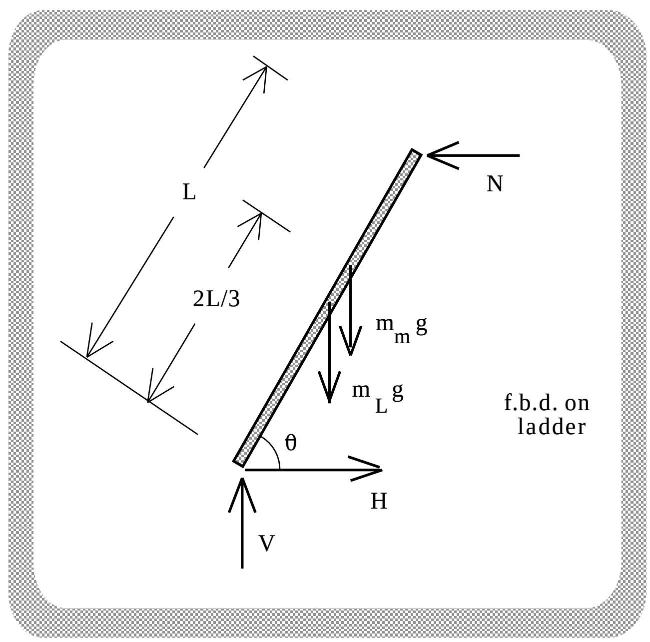


FIGURE 9.6

Solution:

**c.)** All the acting forces are in the *x* and *y directions*, so there is no need to worry about breaking forces into their component parts. We begin by writing:

$\underline{\sum F_x:}$

$$-N + H = (m_m + m_L)\, a_x \qquad (= 0 \text{ as } a_x = 0)$$
$$\Rightarrow \quad N = H.$$

$\underline{\sum F_y:}$

$$-m_m g - m_L g + V = (m_m + m_L)\, a_y \qquad (= 0 \text{ as } a_y = 0)$$
$$\Rightarrow \quad V = (m_m + m_L)\, g.$$

**d.)** We have three unknowns and two equations. The final equation comes from *summing the torques* about any point we choose. For the sake of amusement, let's choose the ladder's center of mass. Using $r_\perp$:

$\underline{\sum \Gamma_{cm}:}$

$$N\,[(L/2)\sin\theta] - m_m g\,[(L/6)\cos\theta] - V\,[(L/2)\cos\theta] + H\,[(L/2)\sin\theta] = I_{cm}\alpha$$
$$= 0 \text{ as } \alpha = 0$$

$$\Rightarrow \quad N = \big[(m_m g/6)\cos\theta + (V/2)\cos\theta - (H/2)\sin\theta\big] / \big[(1/2)\sin\theta\big]$$

**Note 1:** The equation we have generated above has four torque calculations instead of five--the torque due to the weight of the beam produces no torque about the *center of mass* as that force acts *through* the center of mass. *Forces that act through the point about which the torque is being taken produce no torque about that point.*

**Note 2:** This last equation is comprised of *three* unknowns. To solve it, we will have to go back to our original two equations, lift the derived values for *V* and *H* (in terms of *N*) and plug them into this last equation. The end result will be a very messy equation to solve. Once *N* is found, we will then have to go back, plug *N* into the *H* and *V* equations and solve some more.

It would have been so much easier to have generated a "final" equation that had only one unknown in it (say, *N*). *We could have done just that if we had summed the torques about the floor!*

Doing so yields:

$\underline{\sum \Gamma_{floor}:}$

$$N\,[L \sin\theta] - m_m g\,[(2L/3)\cos\theta] - m_L g\,[(L/2)\cos\theta] = I_{floor}\,\alpha = 0$$
$$\Rightarrow \quad N = [(m_m g)\,(2/3)\cos\theta + (m_L g)\,(1/2)\cos\theta]\,/\,\sin\theta .$$

This is a smaller equation (you had to do torque calculations for only three forces) and has only *one* unknown.

Bottom Line: Take your torques about whichever point will eliminate as many unknowns as possible (assuming you don't eliminate them all).

## C.) Rotational Analog to Newton's Second Law:

**1.)** Just as a *net force* motivates a body to *translationally accelerate*, a *torque* motivates a body to *angularly accelerate*. For translational motion, Newton's Second Law states that the sum of the forces acting in a particular direction will equal the *mass* of the object times the object's *acceleration.* Mathematically, this takes the form:

$\underline{\sum F_x:}$

$$(F_{1,x}) \pm (F_{2,x}) \pm (F_{3,x}) \pm \ldots = \pm m a_x .$$

For rotational motion, Newton's Second Law states that the *sum of the torques* acting about any point must equal the *moment of inertia* (the mass-related rotational inertia term) about an appropriate axis through that point times the object's *angular acceleration*. Mathematically, this looks like:

$\underline{\sum \Gamma_p:}$

$$(\Gamma_{F_1}) \pm (\Gamma_{F_2}) \pm (\Gamma_{F_3}) \pm \ldots = \pm I_p\,\alpha$$

The easiest way to see the consequences of this law is by using it in a problem.

**2.)** Example: Determine the angular acceleration $\alpha$ of the beam shown in Figure 9.7a (next page). Assume you know its length *L*, its mass *m*, and the fact that the moment of inertia of a beam about its *center of mass* is $(1/12)mL^2$.

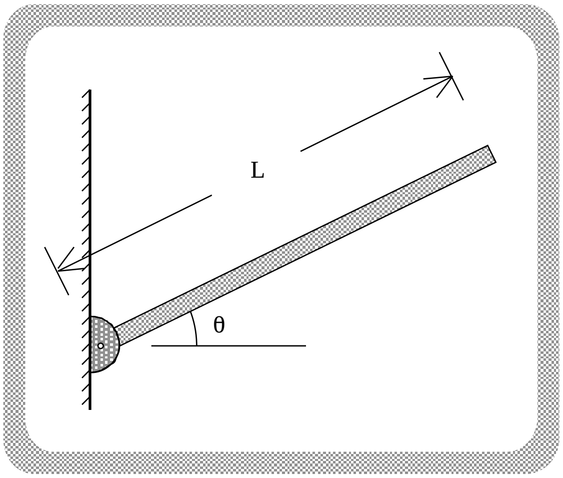


FIGURE 9.7a

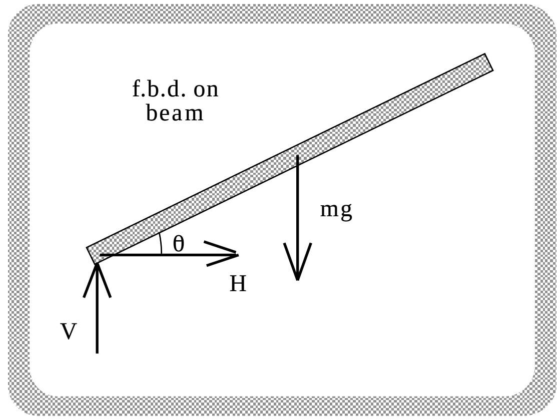


FIGURE 9.7b

**a.)** Using the f.b.d. shown in Figure 9.7b and the Parallel Axis Theorem, the sum of the torques about the axis of rotation (i.e., the pin) is:

$$\underline{\sum \Gamma_{pin}:}$$

$$\begin{aligned} -mg\,(L/2)\cos\theta &= -\,I_{pin}\,\alpha \\ &= -\,[\quad I_{cm} \quad + \quad md^2 \quad]\,\alpha \\ &= -\,[(1/12)\,mL^2 + m(L/2)^2]\,\alpha \\ \Rightarrow \quad \alpha &= 3\,(g\cos\theta)/(2L). \end{aligned}$$

**b.)** Notice that the *angular acceleration* is a function of the beam's angular position $\theta$. As $\theta$ changes while the beam rotates, the *angular acceleration* changes. <u>Conclusion</u>: If you had been asked to determine, say, the *angular velocity* of the beam at some later point in time, you would NOT be able to use rotational kinematics to solve the problem.

## D.) Rotation And Translation Together--Newton's Second Law:

**1.)** Determine the *acceleration* of the hanging mass shown in Figure 9.8 if it is released and allowed to accelerate freely. Assume you know the mass of the hanging

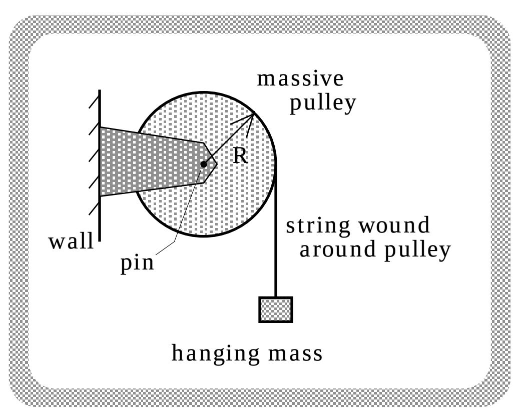


FIGURE 9.8

weight $m_h$, the pulley's mass $m_p$, radius $R$, and the moment of inertia about its *center of mass* $I_{cm} = (1/2)\, m_p R^2$ (we are taking the pulley to be a uniform disk).

**a.)** We are looking for an *acceleration*. This should bring N.S.L. to mind almost immediately. Using that approach with the f.b.d. shown in Figure 9.9 yields:

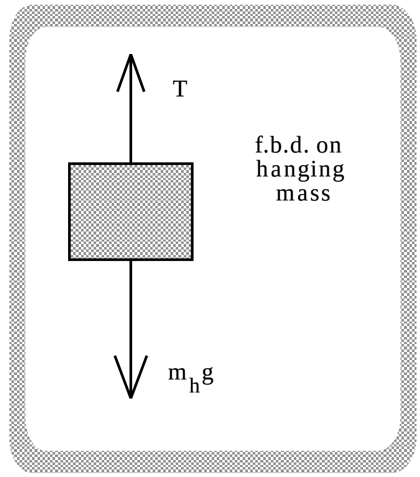


FIGURE 9.9

$\underline{\sum F_y}$:

$$T - m_h g = -m_h a$$
$$\Rightarrow \quad T = m_h g - m_h a \qquad \text{(Equ. A)}.$$

There are two unknowns in this equation, $T$ and $a$. We need another equation.

**b.)** Figure 9.10 shows a *free body diagram* for the forces acting on the pulley. Summing the torques about the pulley's pin yields:

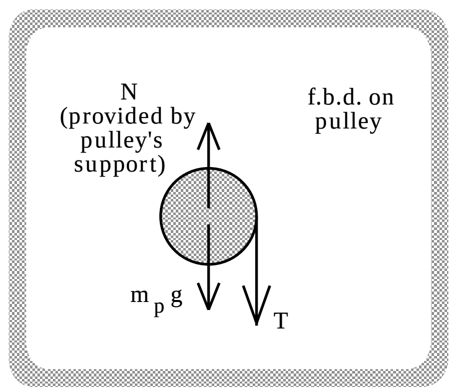


FIGURE 9.10

$\underline{\sum \Gamma_{cm}}$:

$$- TR = - I_{cm}\, \alpha$$
$$= - [(1/2)\, m_p R^2]\, \alpha$$
$$\Rightarrow \quad T = [(1/2)\, m_p R]\, \alpha.$$

**c.)** We can further simplify this equation by remembering that the *translational acceleration a* of a point on the PULLEY'S EDGE a distance $R$ units from the axis of rotation (this will also be the *acceleration of the string*) is related to the *angular acceleration* $\alpha$ of the pulley by:

$$a = R\, \alpha.$$

Using this to eliminate the $\alpha$ yields:

$$T = [(1/2)\, m_p R]\, (a/R).$$
$$= (1/2)\, m_p\, a \qquad \text{(Equation B)}.$$

**d.)** Putting *Equation A* and *Equation B* together yields:

$$(1/2)\, m_p\, a = m_h g - m_h a$$
$$\Rightarrow \quad a = (m_h g)/(m_h + m_p/2).$$

**2.)** A trickier problem: Consider the Atwood Machine shown in Figure 9.11. If the pulley is *massive* and has the same characteristics (i.e., mass, radius, moment of inertia, etc.) as the one used in the problem directly above, determine the *magnitude of the acceleration* of the masses as they freefall. (Assume $m_1$ is more massive than $m_2$).

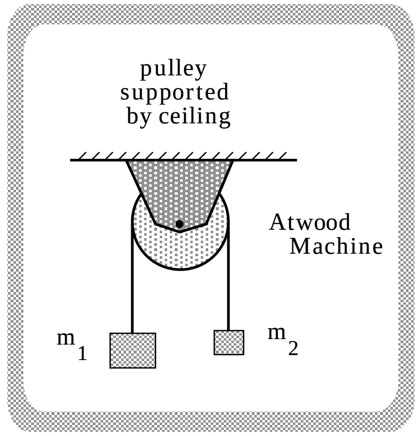


FIGURE 9.11

**Note:** Notice that the f.b.d. in Figure 9.12a depicts a peculiar situation. If both tensions are *T*, the net torque acting about the pulley's *center of mass* is zero. That is, if both tensions are equal, they produce torques that are *equal in magnitude* and *opposite in direction* and, hence, cancel one another out. With no *net torque* acting on the pulley, it will not *angularly accelerate*. And with no angular acceleration, it will not rotate.

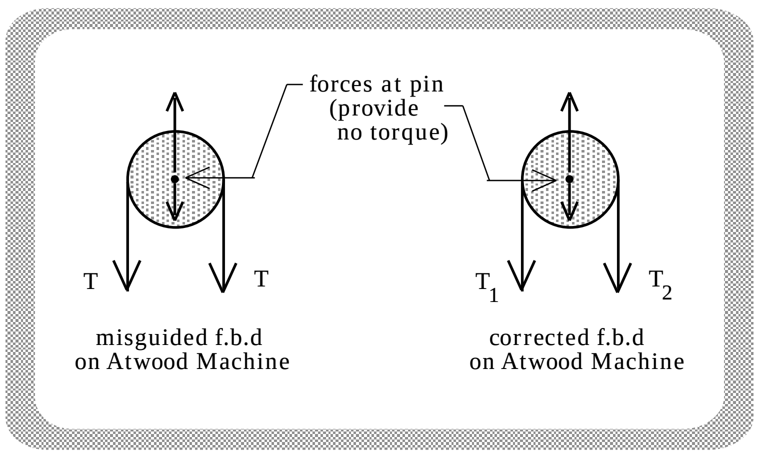


FIGURE 9.12a

FIGURE 9.12b

This clearly is an unacceptable situation--everyone knows that pulleys rotate. The problem? When a pulley is *massive*, the tension in a rope draped over it will not be equal on both sides, assuming the system is angularly accelerating. A more accurate depiction of the forces acting on a massive pulley is, therefore, seen in Figure 9.12b.

**a.)** Figure 9.13a shows the f.b.d. for mass $m_1$:

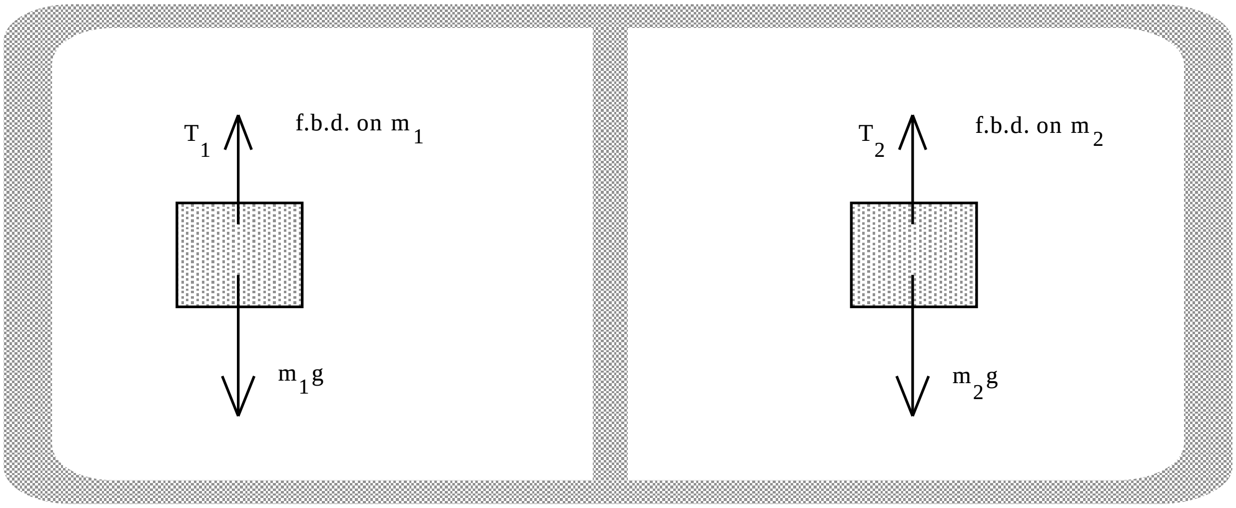


FIGURE 9.13a

FIGURE 9.13b

**b.)** Newton's Second Law yields:

$\underline{\sum F_y}$:

$$T_1 - m_1g = -m_1a$$
$$\Rightarrow \quad T_1 = m_1g - m_1a.$$

Call this *Equation A* (note the sign in front of the acceleration term).

**c.)** Figure 9.13b shows the f.b.d. for mass $m_2$. N.S.L. yields:

$\underline{\sum F_y}$:

$$T_2 - m_2g = m_2a$$
$$\Rightarrow \quad T_2 = m_2g + m_2a.$$

Call this *Equation B*.

**d.)** At this point, we have three unknowns ($T_1$, $T_2$, and $\alpha$) and only two equations. For our third equation, we need to look at the pulley.

**e.)** Figure 9.14 shows the f.b.d. for the pulley. The rotational counterpart of N.S.L. yields:

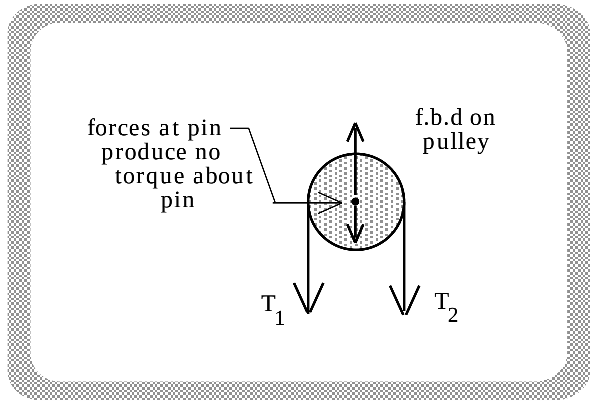


FIGURE 9.14

$\underline{\Sigma \Gamma_{cm}}$:

$$T_1R - T_2R = I_{cm}\,\alpha$$
$$= [(1/2)\, m_pR^2]\;\alpha$$

$$\Rightarrow \quad T_1 - T_2 = [(1/2)\, m_pR]\;\alpha$$

Call this *Equation C* (note the sign in front of the angular acceleration term).

**f.)** *Equation C* introduces another unknown, $\alpha$. Fortunately, we know $\alpha$ in terms of $a$ and $R$:

$$a_{string} = R\;\alpha \qquad \text{(Equation D)},$$

which leaves us with:

$$T_1 - T_2 = [(1/2)\, m_pR]\;(a/R)$$
$$= (1/2)\, m_pa.$$

**g.)** Using Equations A, B, C, and D, we get:

$$T_1 \quad - \quad T_2 \quad = (1/2)\, m_p\, a$$
$$(m_1g - m_1a) - (m_2g + m_2a) = (1/2)\, m_p\, a$$
$$\Rightarrow \quad a = [m_1g - m_2g]/[m_1 + m_2 + m_p/2].$$

**3.)** Rotation with a twist: Consider a hollow ball of radius $R$ and mass $m$ rolling down an incline of known angle $\theta$ (Figure 9.15). What is the *acceleration* of the ball's *center of mass* as the ball rolls down the incline?

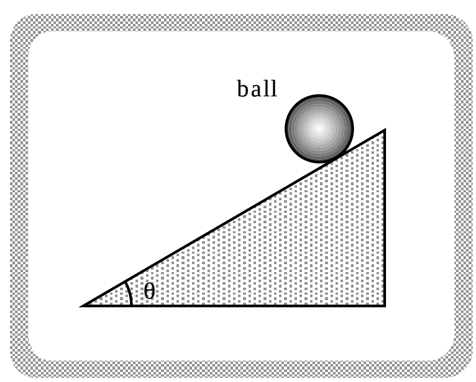


FIGURE 9.15

**a.)** The *free body diagram* for the forces and force-components acting on the ball is shown in Figure 9.16. The axis has been placed along the direction of the *translational acceleration* of the ball's *center of mass*--i.e., along the line of the incline. Noting that there must be *rolling friction* in the system, a summing of the forces along that line yields:

$$\underline{\sum F_x:}$$
$$f_r - mg\sin\theta = -ma_{cm}.$$

Call this *Equation A*.

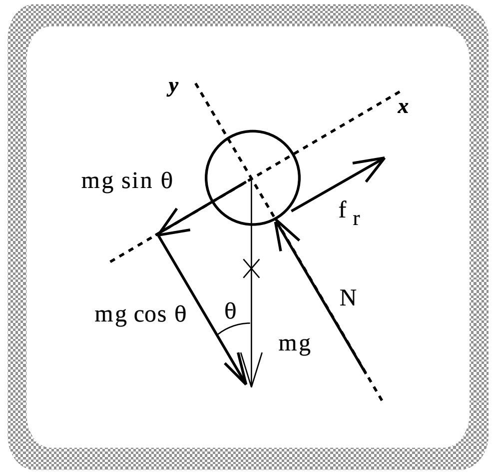


FIGURE 9.16

**b.)** This would be easy if we knew something about the *rolling friction* between the ball and the incline. As we do not have that information, we haven't a clue as to the *magnitude* of the *rolling frictional force* $f_r$.

Stymied, let's consider the rotational counterpart to N.S.L.

**c.)** Noticing that both the *normal force* and the *force due to gravity* pass through the *center of mass*, summing the *torques* about the *center of mass* yields:

$$\underline{\sum \Gamma_{cm}:}$$
$$f_r R = I_{cm}\alpha$$
$$= [(2/3)\ mR^2]\alpha$$
$$\Rightarrow \quad f_r = [(2/3)\ mR]\alpha \qquad \text{(Equation B).}$$

**d.)** We know the relationship between the *acceleration* of the *center of mass* ($a_{cm}$) and the *angular acceleration* about the *center of mass* is:

$$a_{cm} = R\ \alpha \qquad \text{(Equation C).}$$

This means we can write:

$$f_r = [(2/3)\ mR]\ (a/R)$$
$$= (2/3)\ ma.$$

**e.)** Combining *Equations A, B* and *C* yields:

$$f_r \;\; - mg \sin\theta \;\; = - ma_{cm}$$

$$\Rightarrow \quad (2/3)\, ma - mg \sin\theta \;\; = - ma_{cm}$$

$$\Rightarrow \quad a_{cm} = [g \sin \theta]/[1 + (2/3)]$$

$$= (3/5) g \sin \theta .$$

**Note:** Knowing the *translational acceleration* of the center of mass, we can determine the *angular acceleration* of the ball using $a_{cm} = R\alpha$:

$$a_{cm} = R\,\alpha$$

$$\Rightarrow \quad \alpha = a_{cm} / R$$

$$= [(3/5) g \sin \theta] / R.$$

**E.) A Weird But Effective Alternate Approach:**

**1.)** There exists an altogether different way of looking at problems in which a body rolls without slipping. The following endeavors to present the rationale behind the needed perspective.

**2.)** Consider an incline so slippery that a ball is sliding down its face without rolling at all (see Figure 9.17).

**a.)** Relative to the incline, there will be *sliding motion* between the *bottom of the ball* (i.e., the point on the ball that touches the incline--*Point P* in Figure 9.17) and the incline itself.

**b.)** Put another way, at any given instant the point-on-the-ball that happens to be in contact with the incline will have a velocity along the line of the incline (in the *x direction*) relative to the stationary incline.

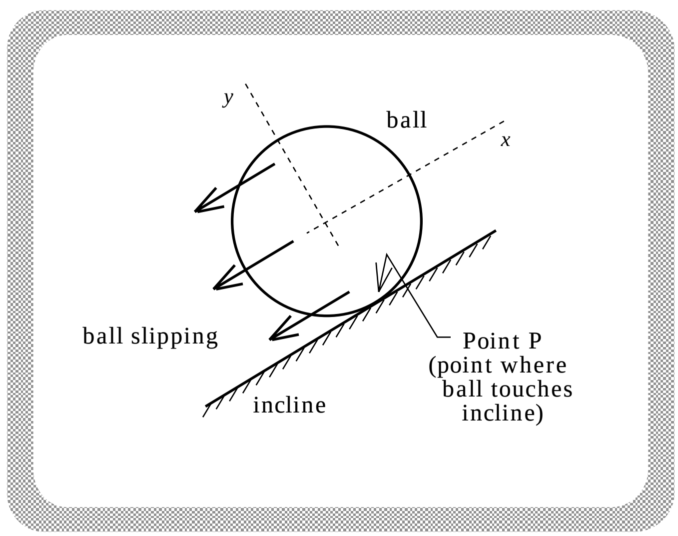


FIGURE 9.17

**c.)** <u>Bottom line</u>: *Point P* moves; the incline does not.

**3.)** A ball that rolls without slipping, on the other hand, will experience *no relative motion* in the *x direction* between *the point that happens to be in contact with the incline* (*Point P* in Figure 9.18) and the STATIONARY incline.

**a.)** This not-so-obvious fact is justified as follows: If the ball is NOT SLIDING over the incline's surface (i.e., as long as we are *not dealing with the case cited in Part 2* above), the velocity of the stationary *incline* and the velocity of the *NON-SLIDING* point-of-contact-with-the-incline must be the same.

**b.)** Bottom line: If the incline's *velocity* in the *x direction* is zero, *Point P's instantaneous velocity* in the *x direction* must also be zero.

**4.)** Let us now take a few moments to re-examine the rolling situation in which there is no slippage between the ball and the incline.

**a.)** Relative to the incline, the *top* of the rolling ball (by *top*, we are talking about the point on the ball farthest from the incline's surface--the point whose *y coordinate* is greatest; this is labeled *Point A* in Figure 9.18) is instantaneously translating faster than the *center of mass* of the ball, and the *center of mass* is instantaneously translating faster than the *bottom* (i.e., *Point P*).

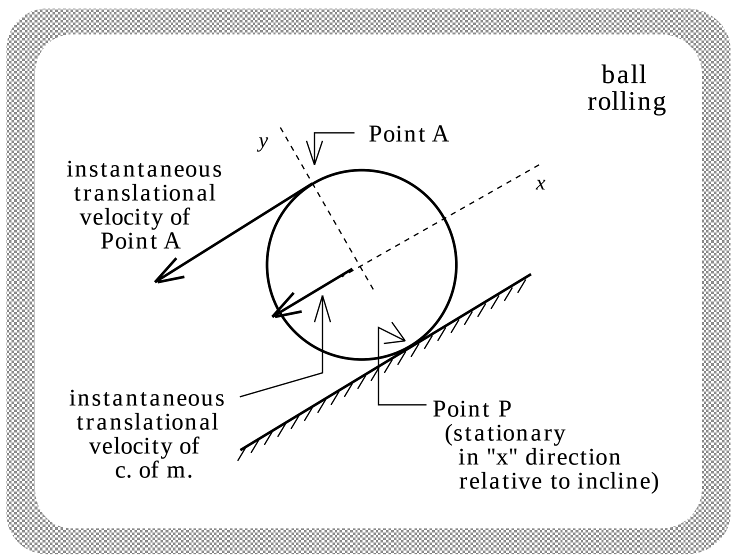


FIGURE 9.18

**b.)** In fact, the *velocity* of *Point A* is twice that of the *center of mass*.

**c. )** As stated above, *Point P* is NOT MOVING AT ALL instantaneously in the *x direction*, relative to the stationary incline (if this isn't clear, ask in class or look at the *Note* at the end of the chapter).

**5.)** Consider another situation. The ball in Figure 9.18 is taken off the incline and a pin is placed through *Point P*. The pin is mounted so that the ball can rotate freely about the pin. The ball is then allowed to freefall. What can we say about the ball as it rotates about an axis through this point on its circumference?

**a.)** Begin by examining Figure 9.19.

**b.)** Notice that *Point A* on the ball is instantaneously translating faster than the *center of mass* of the ball, and the *center of mass* is instantaneously translating faster than the *bottom* (i.e., *Point P*).

This is exactly the same characteristic as was observed in the instantaneous "rolling" situation outlined above.

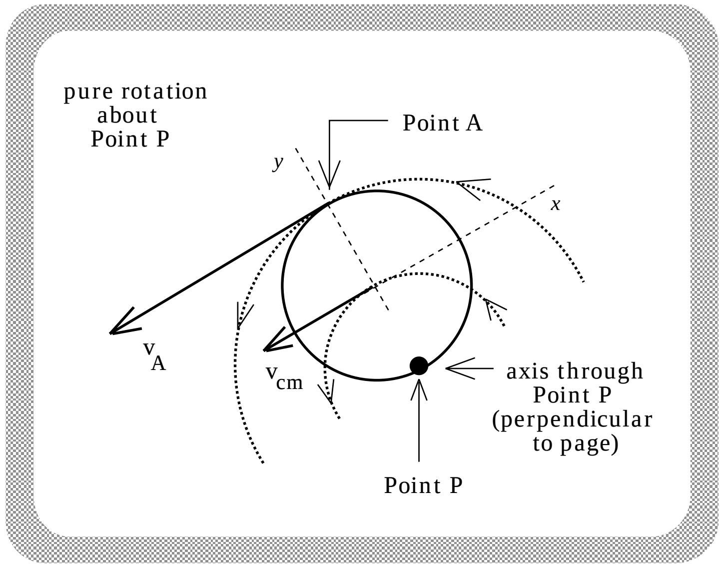


FIGURE 9.19

**c.)** Also, the *instantaneous velocity* of *Point A* is twice that of the *center of mass*.

Again, this is exactly the same characteristic as was observed in the *instantaneous rolling* situation outlined above.

**d.)** Notice that the *translational velocity* of the ball at *Point P* (i.e., at the axis of rotation) is zero.

For the last time, this is exactly the same characteristic that was observed in the *instantaneous rolling* situation outlined above.

**6.)** The *characteristics of motion* for a ball rolling down an incline and a ball pinned to execute a pure rotation appear to be quite similar. In fact, the question arises, "If you could not see what was supporting the ball and only got a quick look, could you be sure which of the two situations you were observing?" That is, could you tell if you were seeing:

**a.)** A ball rolling down an incline (i.e., *rotating about its center of mass* while its *center of mass* additionally *translates* downward toward the left); or

**b.)** A ball executing a pure rotation about an axis through its perimeter?

**c.)** The fact is, if all you got was a glance, it would be *impossible to tell the difference between the two situations*.

**d.)** Consequences: When dealing with a body that is both translating and rotating without slippage (i.e., executing a pure roll), an alternate way to approach the situation is to treat the moving object as though it were instantaneously executing a pure rotation about *its point of contact* with the surface that supports it (*Point P* in the sketch). Analysis to determine, for instance, the "instantaneous acceleration of the center of mass" at a particular instant will yield the same answer no matter which perspective you use. As far as the bottom line goes, they are identical.

**7.)** Do you believe? Let us try both approaches on the same problem and see how the two solutions compare. Reconsider the "ball rolling down the incline" problem.

The question: "What is the *angular acceleration* of a ball of mass *m* as it rolls down the incline shown in Figure 9.20?"

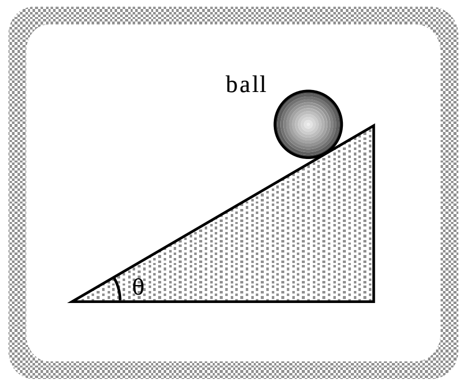


FIGURE 9.20

**Note:** We could as easily have decided to solve for the *instantaneous translational acceleration* of the *center of mass* instead. The two parameters are related by $a_{cm} = R\alpha$; knowing one parameter means we know the other.

**a.)** We have already approached this problem from the point of view of a ball rolling down an incline. The solution for the ball's *angular acceleration* about its *center of mass*, derived in *Part 3e* of the previous section, was:

$$\alpha = [(3/5)g \sin \theta]/R.$$

**b.)** Consider now a *pure rotation about Point P*:

**i.)** From the *free body diagram* shown in Figure 9.21, we begin by summing the torques about *Point P*. As the *normal* and *frictional* forces

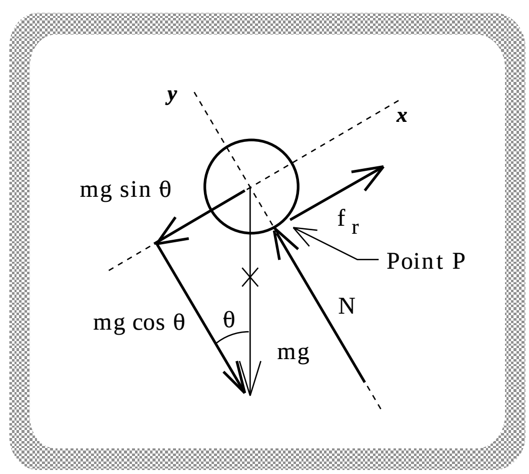


FIGURE 9.21

act through *Point P*, the $r_\perp$ approach yields:

$$\underline{\sum \Gamma_p:}$$
$$(mg)(R \sin \theta) = I_p \alpha.$$

**ii.)** We do not know the *moment of inertia* about *Point P*, only the *moment of inertia* about the *center of mass*. As the torque is being taken about *Point P*, we need $I_p$. Using the *Parallel Axis Theorem*, we write:

$$\begin{aligned} I_p &= I_{cm} + Md^2 \\ &= (2/3)mR^2 + mR^2 \\ &= (5/3)mR^2. \end{aligned}$$

**Note:** The variable *m* in the above equation is the *total mass* in the system (this happens to be the mass of the ball in this case); *d* is the distance between the two parallel axes (i.e., the axis through the *center of mass* and the axis through *Point P*); and *m* is the mass of the ball.

**iii.)** Completing the problem:

$$\underline{\sum \Gamma_p:}$$
$$(mg)(R \sin \theta) = I_p \alpha$$
$$(mg)(R \sin \theta) = [(5/3)\ mR^2]\, \alpha$$

$$\begin{aligned} \Rightarrow \quad \alpha &= [g \sin \theta]/[(5/3)R] \\ &= [(3/5)g \sin \theta]/R. \end{aligned}$$

**iv.)** This is exactly the *angular acceleration* solution we determined using the "rolling" approach to the problem. If we additionally wanted the instantaneous translational acceleration of the *center of mass*, we could use $a_{cm} = R\alpha$, yielding:

$$\begin{aligned} a_{cm} &= R\alpha \\ &= R\ [(3/5)g \sin \theta]/R \\ &= (3/5)g \sin \theta. \end{aligned}$$

Again, this is exactly what we determined using the other method.

**c.)** Bottom line: There are two ways to deal with a rolling object that is not additionally sliding. You can either treat it:

**i.)** As a body executing a pure roll (i.e., as a body whose mass is rotating about its *center of mass* while its *center of mass* is itself translating); or

**ii.)** As a body instantaneously executing a *pure rotation* about its perimeter at the *point of contact* with the structure that supports it.

**8.)** Which way is the best? It depends upon you. The first approach is more conventional but requires the use of both the *translational* and *rotational* counterparts to *Newton's Second Law*. The second approach (the "weird" one) requires only the use of the *rotational* version of *N.S.L.*, but the torques are *not* taken about the *center of mass* so the *parallel axis theorem* must be used to determine the *moment of inertia* $I_p$ about the appropriate axis.

My suggestion is that you use the approach that seems most sensible, given what the system is doing. If, for instance, you see pure rotation, use the rotational approach. If there is rotation and translation happening, use the other approach.

## F.) Energy Considerations and Rotational Motion:

**1.)** Remembering back, energy considerations are useful whenever the forces in a system are conservative and the *velocity* or a *distance traveled* is the parameter of interest. Briefly, energy considerations and the *modified conservation of energy* equation were derived as follows:

**a.)** We began by writing out the WORK/ENERGY THEOREM (i.e., the *net work done* on an object is equal to the *change* of the body's *kinetic energy*) for a body that moves from Position 1 to Position 2 under the influence of a number of forces. The equation was:

$$
\begin{aligned}
W_{net} &= \Delta KE \\
\Rightarrow \quad W_A + W_B + W_C + W_D + \ldots &= \Delta KE,
\end{aligned}
$$

where $W_A$ was the work *force A* did on the object as it moved from Position 1 to Position 2, etc.

**b.)** We derived expressions for the work done by all the *conservative forces with known potential energy functions* as the body moved from Position 1 to Position 2:

$$W_{\text{cons force A}} = - \Delta U_A$$
$$= - ( U_{A,2} - U_{A,1})$$

**c.)** We derived a general expression for the work done by any *non-conservative force*. We did the same for any *conservative forces* for which we did not know a *potential energy function*. For both cases:

$$W_{\text{noPEfct},C} = \mathbf{F}_C \cdot \mathbf{d} \qquad \text{etc.}$$

**d.)** Putting it all into the work/energy theorem (i.e., $W_{net} = \Delta KE$), we ended up with:

$$[-(U_{A,2}- U_{A,1})] + [- (U_{B,2}- U_{B,1})] + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)\, mv_2^{\,2} - (1/2\,)mv_1^{\,2}.$$

**e.)** Rearranging by putting the "before" quantities on the left-hand side of the equation and the "after" quantities on the right-hand side, we got:

$$(1/2)mv_1^{\,2}+ U_{A,1}+ U_{B,1}+ (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^{\,2}+ U_{A,2} + U_{B,2}.$$

**f.)** This was put in short-hand form:

$$KE_1 + \Sigma\, U_1 + \Sigma\, W_{\text{extraneous}} = KE_2 + \Sigma U_2.$$

**g.)** The last touch: We noticed that in a system of more than one body, the *total* kinetic energy in the system at a given instant is the *sum of all the kinetic energies* of all the bodies moving in the system at that instant. As such, the final form of the *modified energy conservation equation* became:

$$\Sigma\, KE_1 + \Sigma\, U_1 + \Sigma W_{\text{extraneous}} = \Sigma\, KE_2 + \Sigma\, U_2.$$

**2.)** This equation also works fine for *rotating* systems. There are only two changes to be made:

**a.)** Although there may still be *kinetic energy* due to the translational motion of bodies within the system, there can now also be *kinetic energy* due to the *rotational* motion of bodies within the system. That means the *total kinetic energy* term has a new member--*rotational kinetic energy*. As such, we need to write:

$$\Sigma\ KE_1 = \Sigma\ KE_{1,trans} + \Sigma\ KE_{1,rot}.$$

**Note:** We determined at the end of the last chapter that just as the *translational kinetic energy* of an object is $(1/2)mv^2$, the *rotational kinetic energy* of an object is $(1/2)(I_{axis\ of\ rot})\omega^2$.

**b.)** Concerning *gravitational potential energy*: Consider a body that moves some *vertical distance h* in a gravitational field. Its change of *potential energy* will be $\underline{+}mgh$. Why? Because the *potential energy function* for *gravity* is related to the *vertical distance traveled*. The question is, "How do you determine the *vertical distance traveled* if the motion is that of a rotating body?"

**i.)** Example: A pinned beam rotates from one angular position to a second (see Figure 9.22 to the right). What is its *change of potential energy* during the move? Put another way, what is *h* in the *mgh* equation that defines changes of *gravitational potential energy*?

**ii.)** As shown in the sketch, *h* is defined as the *vertical displacement* of the body's <u>*center of mass*</u>.

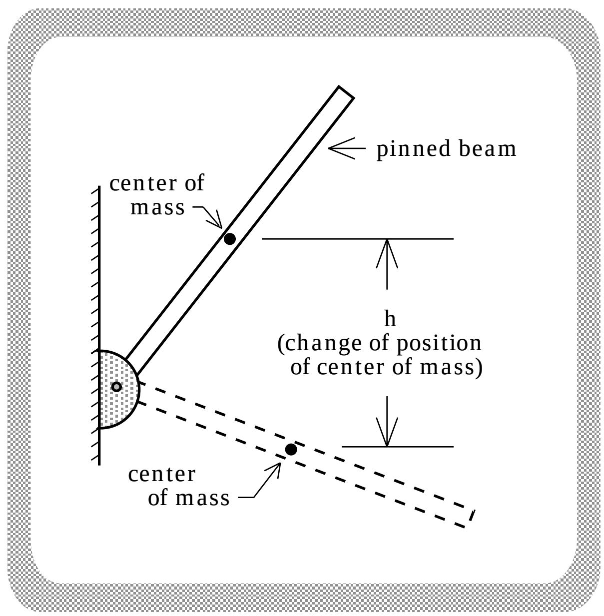


FIGURE 9.22

**Note:** As there is *rotational* <u>*kinetic energy*</u>, is there *rotational* <u>*potential energy*</u>?

The answer to that question is *yes* and *no*. It is theoretically possible to define a *potential energy function* that tells you *how much work* a torque $\Gamma$ does on a body as the body moves through some *angular displacement* $\Delta\theta$, <u>but you will not deal with such a function in this class</u>.

Nevertheless, when you use the *conservation of energy* equation you may be asked to determine the *amount of work* ($\boldsymbol{F}\cdot\boldsymbol{d}$) done as friction acts at the axle of a rotating object. In that case, $d$ equals $r\Delta\theta$, where $r$ is the distance between the *axis of rotation* and the place at which the friction acts.

**G.) Energy Consideration Examples:**

**1.)** A typical <u>*Pure Rotation*</u> Problem: A beam is frictionlessly pinned (see Figure 9.23). From rest, the beam is allowed to freely rotate about its pinned end from an angle $\theta_1 = 30^o$ with the vertical. If the beam's mass is *M*, its length is *L = 2 meters*, and its *moment of inertia* about its *center of mass* is $(1/12)ML^2$, what is its *angular velocity* as it passes through $\theta_2 = 70^o$ with the vertical?

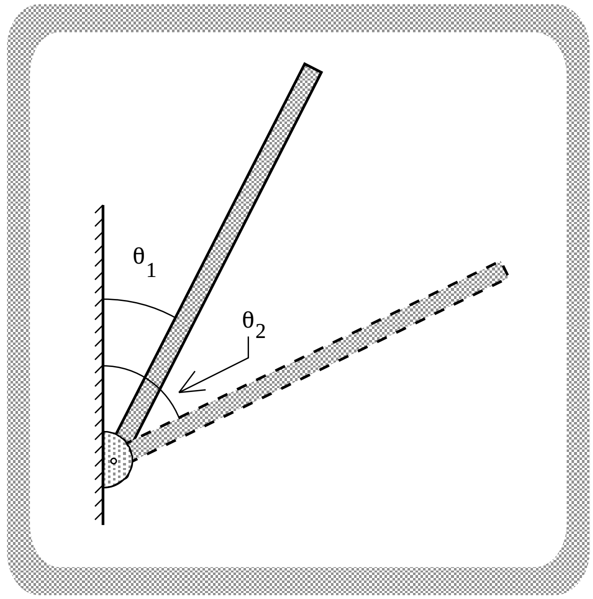


FIGURE 9.23

**a.)** We are looking for a *velocity* (it is an *angular* velocity, but a velocity nevertheless). The first approach that should come to mind whenever a body falls in a gravitational field and a velocity value is requested is the *modified conservation of energy* approach. Executing that approach yields:

$$\begin{array}{ccccccccc} \Sigma\, KE_1 & + & \Sigma\, U_1 & + & \Sigma\, W_{extran} & = & \Sigma\, KE_2 & + & \Sigma\, U_2 \\ 0 & + & Mg\,(L/2)\,(\cos\theta_1 - \cos\theta_2) & + & 0 & = & (1/2)I_p\omega_2^{\ 2} & + & 0\ . \end{array}$$

**Note 1:** Notice that there was no initial *angular velocity*. That didn't have to be the case. DON'T BE LULLED INTO THE BELIEF THAT $v_1$ AND $\omega_1$ WILL *ALWAYS* BE ZERO!

**Note 2:** Figure 9.24 illustrates how the position of the center of mass (depicted by a circle on the beam) changes during the drop. The right triangle used to determine the *final position* of the *c. of m.* relative to the pin is shown in the drawing. A similar triangle would be used to

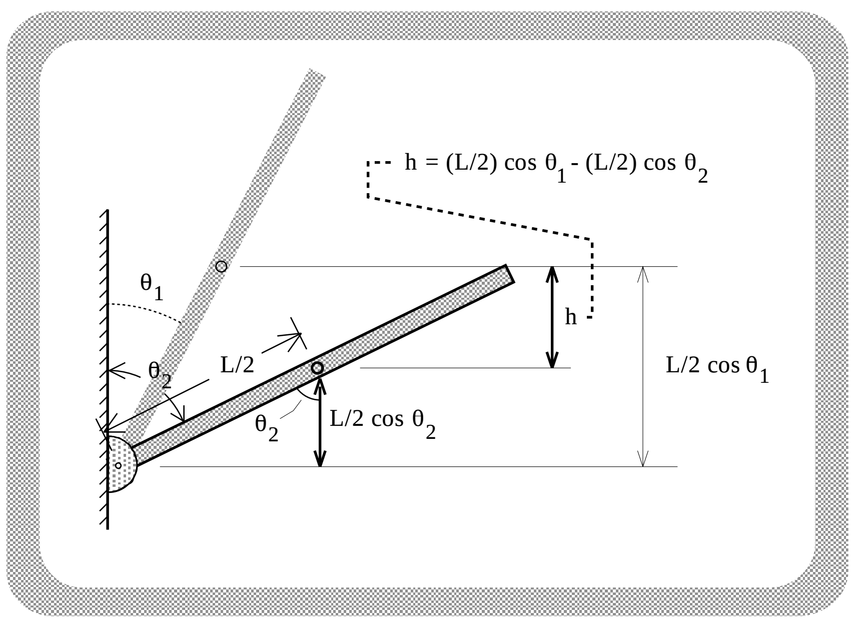


FIGURE 9.24

determine the *c. of m.'s* initial position. The difference between the two yields the drop distance $h$.

**b.)** We know the *moment of inertia* about the *center of mass*. We need the *moment of inertia* about the *pin*. Using the *Parallel Axis Theorem*, we get:

$$I_p = [(1/12)\, ML^2] + M(L/2)^2$$
$$= (1/3)\, ML^2.$$

**c.)** Putting it all together and solving, we get:

$$Mg[(L/2)(\cos 30^o - \cos 70^o)] = (1/2)(ML^2/3)\,\omega_2^{\,2}.$$

**d.)** The *M's* cancel, leaving:

$$(9.8\ m/s^2)(2\ meters/2)(.524) = .5[(2\ meters)^2/3]\,\omega_2^{\,2}$$
$$\Rightarrow \quad \omega_2 = 2.78\ radians/sec.$$

**2.)** A typical <u>*Rotation and Translation*</u> Problem: Consider a string wrapped around a massive pulley. One end of the string is attached to a hanging weight of mass $m$. The system is allowed to accelerate freely. At some instant, the hanging weight is observed to be moving with velocity $v_1 = 3\ m/s$. What will its *velocity* be after it has fallen an additional .8 meters? You may assume that the pulley's mass is $M = 4m$, its radius is $R$, and its *moment of inertia* is $3mR^2$. The system is shown in Figure 9.25.

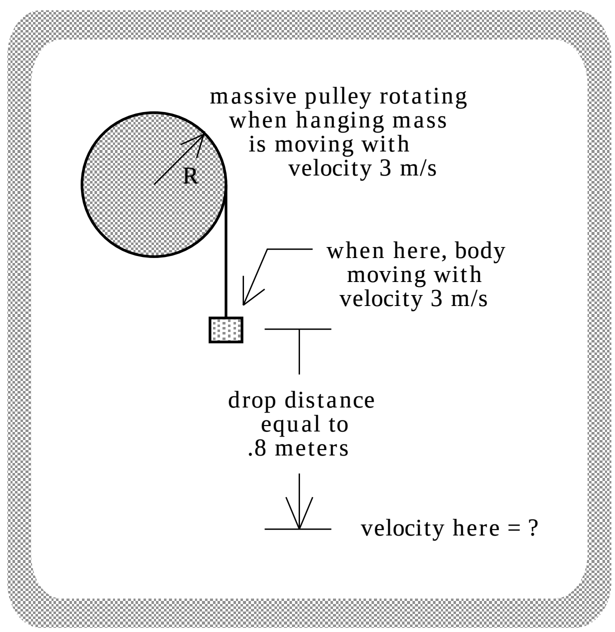


FIGURE 9.25

**a.)** Initially, there is *kinetic energy* wrapped up in both the rotating pulley and the hanging mass, and there is initial *potential energy* wrapped up in the hanging mass. Using the *Conservation of Energy* equation on this situation yields:

$$KE_1 \quad + \sum U_1 \quad + \sum W = \quad \sum KE_2 \quad + \sum U_2$$
$$[.5I_{pul,cm}\omega_1{}^2 + .5mv_1{}^2] + mgh_1 \quad + 0 \quad = [.5\, I_{pul,cm}\omega_2{}^2 + .5\, mv_2{}^2] + [0]$$
$$\Rightarrow \quad .5\,(3mR^2)\,\omega_1{}^2 + .5mv_1{}^2 + mgh_1 = .5\,(3mR^2)\,\omega_2{}^2 + .5\, mv_2{}^2\ .$$

**Important Note:** Why not include *tension in the line* in the work calculation? The short answer: Because it's an internal force. The long answer: Because the work that tension does on $m$ is $-Th$ while the work that tension does on the pulley is $+T(R\Delta\theta) = +Th$. The consequence of all of this is that the net work done by the tension $T$ (again, an internal force) is ZERO.

**b.)** We know that the velocity $v$ of the string (hence the velocity of the hanging mass) is the same as the velocity of the edge of the pulley. This equals $R\omega$. That means $\omega = v/R$. Using this, we can cancel the $m$'s, eliminate the $w$ terms and solve:

$$.5\,(3mR^2)(v_1/R)^2 + .5mv_1{}^2 + mgh_1 = .5\,(3mR^2)(v_2/R)^2 + .5mv_2{}^2$$
$$1.5v_1{}^2 + .5v_1{}^2 + gh_1 = 1.5v_2{}^2 + .5v_2{}^2$$
$$2v_1{}^2 + gh_1 = 2v_2{}^2$$

$$\begin{aligned} \Rightarrow \quad v_2 &= [v_1{}^2 + gh_1/2]^{1/2} \\ &= [(3\text{ m/s})^2 + (9.8\text{ m/s}^2)(.8\text{ meters})/2]^{1/2} \\ &= 3.59\text{ m/s}. \end{aligned}$$

**c.)** The question could as easily have asked for the *final angular velocity* of the pulley. It is the same problem with one exception: you would have eliminated the $v$ terms with $v = R\omega$ instead of eliminating $\omega$ with $\omega = v/R$.

If we knew beforehand that $v_2 = 3.59\ m/s$, we would have used $v = R\omega$ to calculate:

$$\begin{aligned} \omega_2 &= v_2/R \\ &= (3.59\text{ m/s})/R. \end{aligned}$$

**3.)** A typical *Rotation and Translation Mixed In One* Problem: Figure 9.26 shows a ball rolling *up* a $30^o$ incline. At the initial instant, the ball's *center of mass* is moving with velocity $v_1 = 8\ m/s$. How fast will its

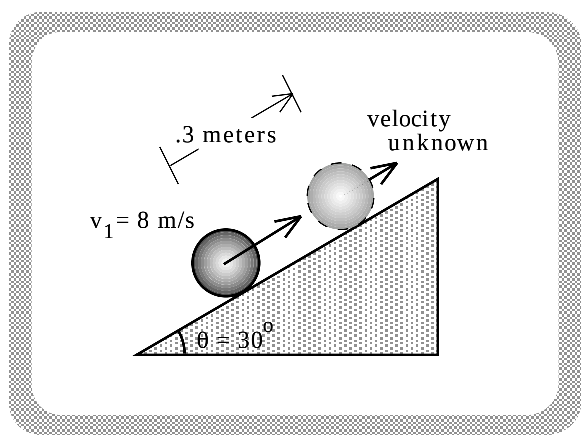

FIGURE 9.26

*center of mass* be moving after traveling an additional .3 meters up the incline? Assume the ball's mass is $m = .2\ kg$, its radius is $R = .1\ meters$, and its *moment of inertia* about its *center of mass* is $(2/5)mR^2$.

**Note:** Thinking back to the section on *angular acceleration* and Newton's Second Law, we found that any situation in which an object *rolls without slipping* can be treated either as: 1.) motion around the *center of mass* plus motion of the *center of mass* (i.e., a roll), or 2.) an instantaneous, *pure rotation* about the point of contact between the object and the support upon which it rolls (i.e., a pure rotation). We will approach this problem both ways.

The "*rotation* and *translation* of the center of mass" approach:

**a.)** Looking at the ball's motion when first observed, two *kinds* of motion are taking place relative to the ball's *center of mass*. The ball is rotating around its *center of mass* with angular velocity $\omega_1$, and the ball's *center of mass* is itself moving with velocity $v_1$. In other words, the initial kinetic energy, as far as the center of mass of the system is concerned, is:

$$KE_{1,tot} = (1/2)I_{1,cm}\,\omega_1^{\ 2} + (1/2)mv_1^{\ 2}.$$

**b.)** For the sake of ease, let us define the *gravitational potential energy* of the ball when at *Position 1* as zero.

**c.)** (THIS IS IMPORTANT): Rolling friction exists within the system, but rolling friction does so little work on the ball that the energy loss due to it is negligible. As such, we will approximate it to be zero. That means that there are no extraneous forces doing work on the system which, in turn, means that $\sum W_{extr} = 0$.

**d.)** Writing out the *conservation of energy* equation, we get:

$$\begin{array}{ccccccc} \Sigma\ KE_1 & + & \Sigma\ U_1 & + & \Sigma\ W_{ext} & = & \Sigma\ KE_2 & + \Sigma\ U_2 \\ \{(1/2)I_{1,cm}\,\omega_1^{\ 2} + (1/2)mv_1^{\ 2}\} & + & 0 & + & 0 & = & \{(1/2)I_{2,cm}\,\omega_2^{\ 2} + (1/2)mv_2^{\ 2}\} & + mgh, \end{array}$$

where $h$ is the vertical rise of the ball's center of mass.

**e.)** We know that $v_{cm} = R\omega$. Using that and substituting $I_{cm}$ *for a ball* into the equation yields:

$$.5\ [(2/5)mR^2]\ (v_1/R)^2 + .5\ mv_1^{\ 2} = .5\ [(2/5)mR^2](v_2/R)^2 + .5mv_2^{\ 2} + \ mgh$$

$$(1/5)mv_1^{\ 2} + .5mv_1^{\ 2} = (1/5)mv_2^{\ 2} + .5mv_2^{\ 2} + mgh$$

$$(7/10)mv_1^{\ 2} = (7/10)mv_2^{\ 2} + mgh$$

$$\Rightarrow \quad v_2 = [[.7v_1^{\ 2} - gh_2]/(.7)]^{\ 1/2}$$

$$= [v_1^{\ 2} - 1.43gh]^{1/2}.$$

**f.)** To find *h*, we need to use trig to determine the VERTICAL distance traveled as the ball rolled .3 meters up the incline. We know that the definition of the *sine* of an angle is equal to "the side opposite the angle divided by the hypotenuse." In this case, the "opposite side" is *h* and the hypotenuse is .3 meters. Using this, we get:

$$h = (.3\text{ meters})\ (\sin 30^o)$$

$$= .15\text{ meters}.$$

**g.)** Putting in the numbers, we get:

$$v_2 = [(8\text{ m/s})^2 - 1.43(9.8\text{ m/s}^2)(.15\text{ meters})]^{1/2}$$

$$= 7.87\text{ m/s}.$$

<u>The "*pure rotation*" approach:</u>

**h.)** Reiterating what has previously been stated, we know that if we look at an object's instantaneous motion, we can't tell whether the object is *rolling* or moving in *pure rotation* about a point on its perimeter. We've analyzed the "conservation of energy" problem outlined above from the first perspective. Now we will deal with the problem using the "pure rotation" approach.

**i.)** The sketch in Figure 9.27 (next page) assumes the ball is executing a pure rotation (instantaneously) about a *Point P* located at the intersection of the ball and the incline. If we take the *angular velocity* of the ball about P at that instant to be $\omega_1$, the ball's initial *kinetic energy* will be purely rotational about *Point P* and will equal:

$$KE_{1,tot} = (1/2)I_{1,p}\omega_1^{\,2}.$$

A similar expression defines the ball's *kinetic energy* after traveling up the incline .3 meters.

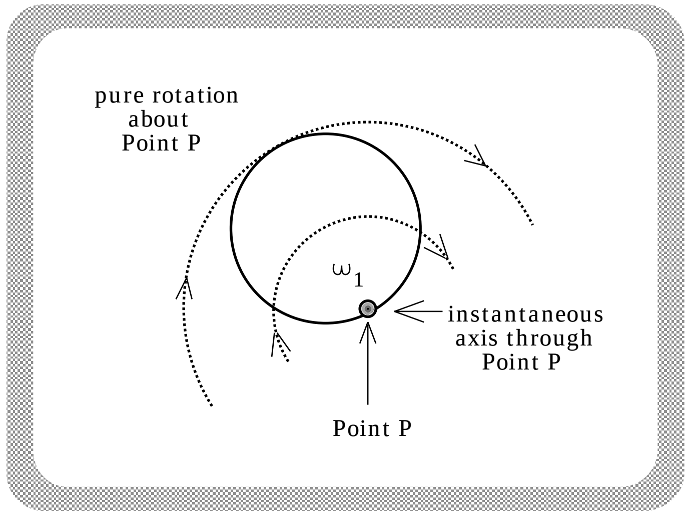


FIGURE 9.27

**j.)** For the sake of ease, let us define the *gravitational potential energy* of the ball at *Position 1* as zero.

**k.)** Writing out the *conservation of energy* equation, we get:

$$\begin{array}{ccccccc} \Sigma\, KE_1 & + & \Sigma\, U_1 & + & \Sigma\, W_{ext} & = & \Sigma\, KE_2 + \Sigma\, U_2 \\ (1/2)I_{1,p}\omega_1^{\,2} & + & 0 & + & 0 & = & (1/2)I_{2,p}\omega_2^{\,2} + mgh. \end{array}$$

**l.)** We know the *moment of inertia* about the *center of mass*; we need the *moment of inertia* about the *pin*. Using the *Parallel Axis Theorem*, we get:

$$\begin{aligned} I_p &= [(2/5)\, mR^2] + mR^2 \\ &= (7/5)mR^2. \end{aligned}$$

**m.)** Substituting and solving yields:

$$\begin{aligned} (1/2)I_{1,p}\omega_1^{\,2} &= (1/2)I_{2,p}\omega_2^{\,2} + mgh \\ (1/2)\,[(7/5)\, mR^2]\,\omega_1^{\,2} &= (1/2)[(7/5)\, mR^2]\,\omega_2^{\,2} + mgh \\ (7/10)\, mR^2\omega_1^{\,2} &= (7/10)mR^2\omega_2^{\,2} + mgh \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \omega_2 &= [.7R^2\omega_1^{\,2} - gh]/(.7R^2)]^{1/2} \\ &= [\omega_1^{\,2} - 1.43gh/R^2]^{1/2}. \end{aligned}$$

**n.)** To determine the velocity of the *center of mass,* we will have to use $v_{cm} = R\omega$. Doing so yields:

$$\omega_2 = [\omega_1^2 - 1.43\ gh/R^2]^{1/2}$$
$$\Rightarrow\quad v_{cm} = R\,[\omega_1^2 - 1.43\ gh/R^2]^{1/2}$$
$$= [R^2\omega_1^2 - R^2(1.43\ gh/R^2)]^{1/2}$$
$$= [v_1^2 - 1.43\ gh]^{1/2}.$$

As expected, our solutions from the two approaches are the same.

**Note:** WHICH APPROACH IS BEST? It depends upon the problem. The first requires more terms in the *conservation of energy* equation; the second utilizes a simpler form of the *conservation of energy* equation but requires the use of the *parallel axis theorem*.

My suggestion? Learn both approaches and use whichever seems easiest for a given problem.

**H.) Comments on Test Questions: N.S.L. and ENERGY Considerations:**

**1.)** When you are asked to determine an *acceleration* or *angular acceleration*, the first approach you should consider is *Newton's Second Law*. It won't always work, but it is one of the most powerful acceleration-involved approaches available to you.

When you are asked to determine a *velocity* or *angular velocity* in a non-collision situation, the first approach you should consider is *conservation of energy*. Again, it will not always work but it is a very powerful approach.

**2.)** A typical test question will have a number of parts to it. You could, for instance, be given a ball rolling down an incline and be asked to:

**a.)** Derive an expression for the acceleration of the system;

**b.)** Derive an expression for the velocity of the ball after having rolled down the incline a distance *h*;

**c.)** Determine the angular velocity of the ball at the point defined in *Part b*.

**3.)** You no longer have the cues available in previous chapters (i.e., you can no longer assume that because the chapter you are studying is, for instance, about Newton's Second Law, that the test problems will be Newton's Second Law problems only). You must now first identify the *kind* of problem you are looking at, then have the wherewithal to use the appropriate approach.

## I.) Conservation of Angular Momentum:

**1.)** Just as a body moving in straight-line motion has *momentum* defined as the product of its inertia (its *mass*) and its *velocity*, a rotating body has *angular momentum* defined as the product of its rotational inertia (its *moment of inertia*) and its *angular velocity*. Mathematically, these two are:

$$\mathbf{p} = m\mathbf{v} \qquad \text{and} \qquad \mathbf{L} = I\boldsymbol{\omega}.$$

**Note:** Both *momentum* and *angular momentum* are vectors. As you will never have to worry about two or three-dimensional *angular momentum*, the only part of the vector notation you will normally use when writing out an *angular momentum* quantity is the sign. An *angular momentum* is considered "+" if it is associated with motion that is *counterclockwise* relative to the point about which the *angular momentum* is calculated (if this is a pure rotation, positive *angular momentum* would correspond to positive *angular velocity*). Negative *angular momentum* is just the opposite.

**2.)** Newton observed that there exists a relationship between the *net force* acting on a body and the body's *change of momentum*. In one dimension, that relationship is:

$$\mathbf{F}_{net} = d\mathbf{p}/dt$$

or, if the force is constant and the time interval large,

$$\mathbf{F}_{net} = \Delta\mathbf{p}/\Delta t.$$

A similar relationship exists between the net torque acting on a body and the body's change of *angular momentum*. That relationship is:

$$\boldsymbol{\Gamma}_{net} = d\mathbf{L}/dt$$

or, if the torque is constant and the time interval large,

$$\boldsymbol{\Gamma}_{net} = \Delta\mathbf{L}/\Delta t.$$

**Big Note:** If the sum of the *net external torque* is zero, the *CHANGE* of the system's *ANGULAR MOMENTUM* will be ZERO and the *ANGULAR MOMENTUM* will be CONSERVED.

**3.)** In dealing with torque calculations, we found that there are two general ways to determine the net torque being applied to a body.

**a.)** Using strictly *translational* variables, we write:

$$\mathbf{\Gamma}_{net} = \mathbf{r} \times \mathbf{F}_{net}.$$

**b.)** Using strictly *rotational* variables, we write:

$$\mathbf{\Gamma}_{net} = I\boldsymbol{\alpha}.$$

**4.)** Analogous to the *torque* situation, there are two general ways to determine the *angular momentum* of a body:

**a.)** Using strictly translational variables (instantaneous), we write:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

**b.)** Using strictly rotational variables we write:

$$\mathbf{L} = I\boldsymbol{\omega}.$$

**5.)** Bottom line: There are two ways to determine the *angular momentum* of a point mass. If you know the *moment of inertia* of the body about its axis of rotation and its *angular velocity*, you can use $L = I\omega$ (this also works for extended objects). If you know the body's *instantaneous momentum* (mv) and a position vector $r$ that defines its position relative to the axis of rotation, you can use the relationship $L = |\mathbf{r} \times \mathbf{p}|$.

**a.)** Example: Determine the *angular momentum* of an object of mass $m$ circling with velocity magnitude $v$ and angular velocity $\omega$ a distance $R$ units from the axis of rotation (see sketch in Figure 9.28).

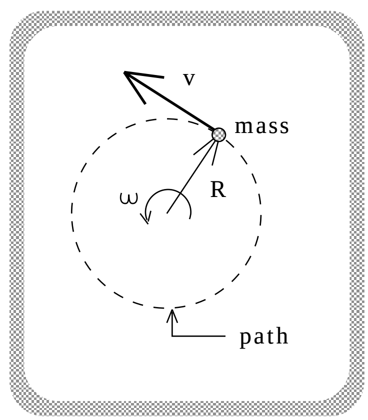

FIGURE 9.28

**i.)** The rotational relationship: Noting that the *moment of inertia* of a point mass a distance $R$ units from the axis of rotation is $mR^2$, the magnitude of the angular momentum is:

$$\begin{aligned} L &= I\omega \\ &= (mR^2)\,\omega. \end{aligned}$$

**ii.)** The translational relationship: Noting that the magnitude of the instantaneous momentum of the body is $p = mv$, and that the angle between the line of $\boldsymbol{r}$ and the line of $\boldsymbol{p}$ is $90^o$, we have:

$$\begin{aligned} L &= |\mathbf{r} \times \mathbf{p}| \\ &= r\,(mv \sin 90^o) \\ &= mvR. \end{aligned}$$

Noting additionally that $v = R\omega$, we can write:

$$\begin{aligned} L &= mvR \\ &= m\,(R\omega)\,R \\ &= m\,R^2\,\omega. \end{aligned}$$

In both cases, the body's *angular momentum* is the same.

**6.)** Earlier, it was pointed out that when the net torque acting on a body equals zero, $\Gamma_{net} = \Delta L/\Delta t = 0$. This implies the angular momentum $L$ does not change with time (i.e., $L$ is constant). An expanded way of stating this is embodied in the *conservation of angular momentum* equation. Analogous to the *modified conservation of momentum* equation, this relationship for one dimensional rotational motion (i.e., rotational about a fixed axis) is written as:

$$\sum L_1 + \sum\,(\Gamma_{ext}\,\Delta t) = \sum L_2.$$

**a.)** This relationship states that in a particular direction, the *sum* of the *angular momenta* of all the pieces of a system at time $t_1$ will equal the sum of all of the *angular momenta* at time $t_2$ if there are *no external torques* acting on the system to change the net *angular momentum* during the time period. If *external torques* do exist, the final *angular momentum* increases or decreases during the time period by $\sum(\Gamma_{ext}\,\Delta t)$.

**b.)** When the $\sum(\Gamma_{ext}\,\Delta t)$ term is zero, angular momentum is said to be *conserved*. This occurs either when there are *no external torques* acting on the system or when *external torques* present are so small and/or act over such a tiny $\Delta t$ that to a good approximation they do not appreciably alter the system's motion (hence, the system's total angular momentum).

**c.)** The most common use of the *conservation of angular momentum* is in the analysis of collision problems (explosion problems, for instance, are nothing more than fancy collision problems). Freewheeling collisions happen so quickly that even if there are *external torques* acting on the system, the total angular momentum of the system *just before* the collision and *just after* the collision will be the same. In other words, angular momentum is usually conserved *through a collision*.

**7.)** Example #1 (situation in which a system's *moment of inertia* changes but no external torques are applied): An ice skater begins a spin with his arms out. His *angular velocity* at the beginning of the spin is $\omega_1 = 2$ *radians/sec* and his *moment of inertia* is 6 kg·m$^2$. As the spin proceeds, he pulls his arms in, decreasing his moment of inertia to 4.5 kg·m$^2$. What is his *angular velocity* after pulling in his arms?

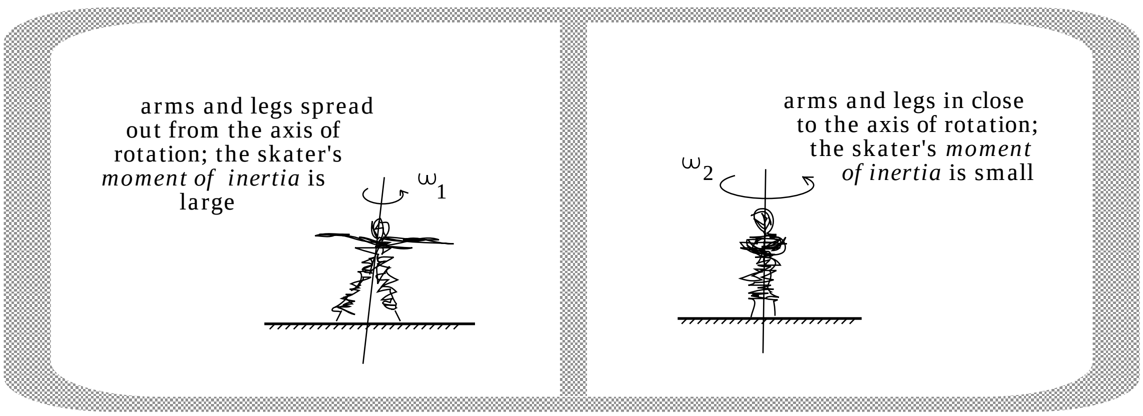


FIGURE 9.29a

FIGURE 9.29b

Solution: Figures 9.29a and 9.29b show the skater before and after pulling in his arms. The work required to do the pulling is provided by the "burning" of chemical energy wrapped up in the muscles of his body. The force applied due to that exertion provides no net torque (not only would any such torque be internal to the system if it existed, there is in fact no torque at all because the *line* of the muscle forces *acts through the axis of rotation*).

As there are no *external torques* being applied to the skater, his *angular momentum* must remain the same throughout (i.e., it is conserved).

**a.)** At the beginning of the spin, his *angular momentum* is:

$$\begin{aligned} L_1 &= I_1 \omega_1 \\ &= (6\ \text{kg}\cdot\text{m}^2)\ (2\ \text{rad/sec}) \\ &= 12\ \text{kg}\cdot\text{m}^2/\text{s}. \end{aligned}$$

**b.)** After his arms are pulled in, his *moment of inertia* decreases and the *angular momentum* expression becomes:

$$L_2 = I_2 \omega_2$$
$$= (4.5\ \text{kg}\cdot\text{m}^2)\ (w_2).$$

**c.)** Equating the two *angular momentum* quantities:

$$L_1 = L_2$$
$$12\ \text{kg}\cdot\text{m}^2/\text{s} = (4.5\ \text{kg}\cdot\text{m}^2)\ (w_2)$$
$$\Rightarrow \quad \omega_2 = 2.67\ \text{rad/sec}.$$

**Note:** Although angular momentum is conserved here, energy is not conserved. The skater has to use chemical energy within his muscles in pulling in his arms. A comparison of the energy before and after the pull-in shows that there is more kinetic energy in the system *after* the pull-in than *before*. (Try it. You should find that $E_1 = 12$ *joules* while $E_2 = 16$ *joules*.)

**8.)** Example #2 (situation in which a system's *moment of inertia* changes but no external torques are applied): A child of mass 40 kg walks from the edge of a 4 meter radius *merry-go-round* (moment of inertia $I_{m.g.r.} = 700\ kg\cdot m^2$) to a position 1.5 meters from the *merry-go-round's* center. If the system initially rotates at 3 radians/second, what is the system's *angular velocity* once the kid reaches the 1.5 meter mark? See Figures 9.30a and 9.30b for "before and after" views.

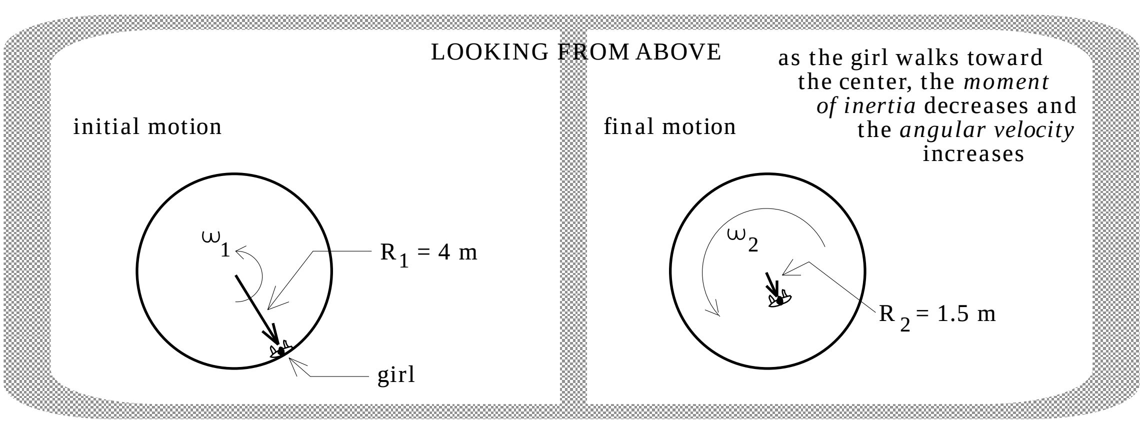


FIGURE 9.30a

FIGURE 9.30b

Solution:

**a.)** Once again, any change in the *angular momentum* of the *merry-go-round* will be due to a *torque* exerted by the walking kid. But according to Newton's Third Law, any torque the kid exerts on the *merry-go-round* must be matched by an equal and opposite torque exerted by the *merry-go-round* on the kid. In other words, there are only *internal torques* acting on the system. This implies that *angular momentum* is conserved.

**b.)** With that in mind:

$$L_{1,tot} = L_{2,tot}$$
$$[L_{1,kid} + L_{1,m.g.r.}] = [L_{2,kid} + L_{2,m.g.r.}]$$
$$[I_{1,kid}\omega_1 + I_{m.g.r.}\omega_1] = [I_{2,kid}\omega_2 + I_{m.g.r.}\omega_2]$$
$$(mR_1^{\,2})\omega_1 + (700\ kg\cdot m^2)\ \omega_1 = (mR_2^{\,2})\ \omega_2 + (700\ kg\cdot m^2)\ \omega_2$$

$$(40\ kg)(4\ m)^2(3\ rad/sec) + (700\ kg\cdot m^2)\ (3\ rad/sec)$$
$$= (40\ kg)(1.5\ m)^2\,\omega_2 + (700\ kg\cdot m^2)\ \omega_2$$
$$\Rightarrow \quad \omega_2 = 5.1\ rad/sec.$$

**Note:** This makes sense. If the *moment of inertia* of the kid decreases as she gets closer to the center of the merry-go-round, the system's *angular velocity* must increase if *angular momentum* is to remain constant.

**9.)** Example #3 (situation in which angular momentum is conserved through a collision): A child of mass *m* runs clockwise with velocity $v_1$ right next to a merry-go-round of mass *M,* radius *R*, and moment of inertia $.5MR^2$ (i.e., the child's radius of motion is effectively *R*). The merry-go-round is moving counterclockwise with angular velocity $\omega_1$, where $\omega_1$ is not related to $v_1$. The child jumps on at the merry-go-round's edge. What is the final velocity of the child?

**a.)** The torque that changes the child's motion is produced by the child's interaction with the merry-go-round, and the torque that changes the merry-go-round's motion will be produced by its interaction with the child. In other words, the torques in the system will be internal. As such, the *total angular momentum* before the collision and after the collision must be the same.

**b.)** With that in mind:

$$L_{1,tot} = L_{2,tot}$$
$$[L_{1,kid} + L_{1,m.g.r.}] = [L_{2,kid} + L_{2,m.g.r.}]$$
$$[-mv_1R + (.5MR^2)\omega_1] = [mv_2R + (.5MR^2)\omega_2].$$

**Note 1:** We are assuming that the merry-go-round slows with the collision, but that it continues in the counterclockwise (i.e., positive) direction. That means the child reverses direction with the collision.

**Note 2:** The child's initial angular momentum is associated with clockwise motion. As such, the angular momentum is *negative*. If you don't believe me, do ***rxp*** and determine the appropriate sign for the cross product.

As $v_2 = R\omega_2$ (remember, $v_1 \neq R\omega_1$) we can write:

$$[-mv_1R + (.5MR^2)\omega_1] = [mv_2R + (.5MR^2)(v_2/R)].$$

Canceling out $R$ terms and solving, we get:

$$v_2 = [-mv_1 + (.5MR)\omega_1]/[.5M + m].$$

**J.) Simultaneous Sliding and Rolling**:

**1.)** There is a class of problems characterized by the fact that within the set-up, partial sliding occurs (that is, sliding and rolling happening at the same time). The difficulty with such problems is that nothing is conserved and, if there is rolling and translating occurring during the slide, there is no known relationship between the body's center of mass velocity and its angular velocity (i.e., $v_{cm} \neq R\omega$). An example will highlight this kind of situation.

**2.)** Example: When a bowling ball is thrown by an accomplished bowler, the ball will often start out with no spin at all (i.e., will initially execute a pure slide), picking up spin as friction between the floor and the ball does work on the ball, making it roll. In such cases, there is a period of time during which the ball partially rolls and partially slides. With that in mind, consider the following scenario: A bowling ball is thrown with an initial velocity of $v_o$, initially executing a pure slide. Almost immediately, it begins to both slide and roll (friction acts immediately). Only after some time does the ball finally execute a pure roll (see Figure 9.31 on the next page).

The question?

**a.)** Justify the statement, "Nothing is conserved during the collision."

**b.)** Determine the ball's velocity $v_1$ when it finally begins to execute a <u>pure roll</u>. This should be in terms of $v_o$.

<u>Solution</u>:

**i.)** *Energy* is not conserved as friction (a non-conservative force) is acting. Furthermore, we can not use the *extended conservation of energy equation* because we don't have enough information to determine the magnitude of the frictional force or, for that matter, the distance over which sliding friction acts.

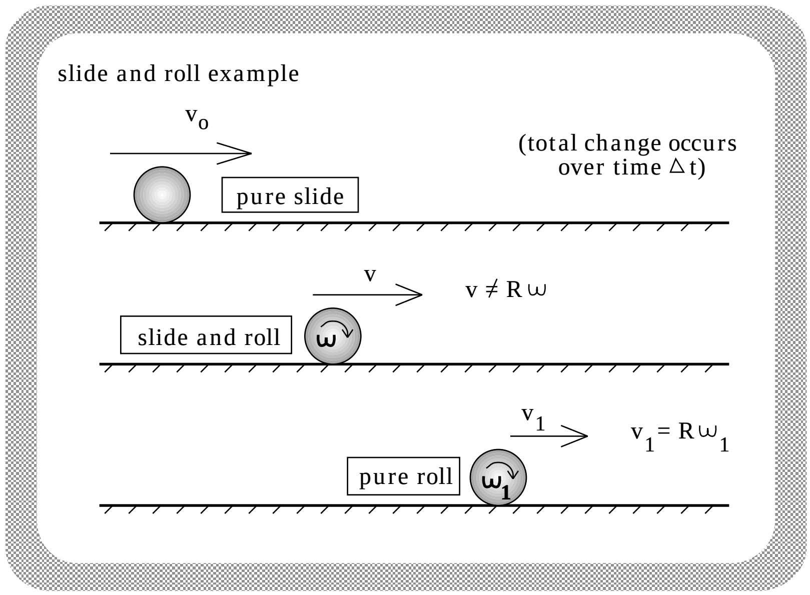


FIGURE 9.31

*Momentum* is not conserved as there is an external force acting on the ball (friction from the floor is not a part of *the system*). Even if that isn't obvious, the initial momentum is $mv_o$ while the final momentum is $mv_2$. As the problem is stated, the two *cannot* be the same.

*Angular momentum* is not conserved as there is an external torque acting on the ball due to friction (in fact, there *is* no initial angular momentum as the ball starts out with a pure slide).

**ii.)** Because we can't use any of the conservation theorems, we must go back to first principles, which is to say Newton's Second Law (see f.b.d. in Figure 9.32). If we assume an average frictional force $f_{avg}$ acts on the ball over the time it takes for the ball to go from a pure slide to a pure roll, and if we assume that time equals $\Delta t$,

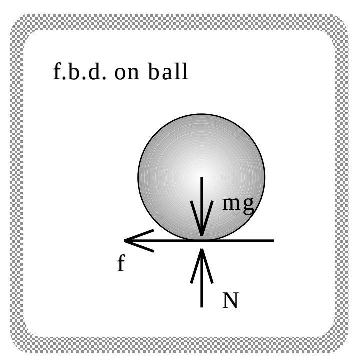


FIGURE 9.32

we can express the acceleration in terms of *velocity* and *time* and write:

$$\underline{\sum F_x:}$$
$$-f_{avg} = ma$$
$$= m[(v_1 - v_o) / \Delta t]$$
$$\Rightarrow \quad -f_{avg}\Delta t = m(v_1 - v_o).$$

**Note:** As the average force is negative, we would expect a negative sign in front of the acceleration term. That sign was not unembedded in our equation because $v_1$ is less than $v_o$. As such, the difference $v_1 - v_o$ will itself be negative and the required negative will not be lost.

The moment of inertia of a ball is $I_{ball} = (2/5)mR^2$. Summing the torques acting on the ball over the pure-slide to pure-roll time interval (i.e., $\Delta t$), we get:

$$\underline{\sum \Gamma_{cm}:}$$
$$-(f_{avg}R) = -I\alpha$$
$$= -[(2/5)mR^2][(\omega_1 - 0) / \Delta t]$$
$$\Rightarrow \quad f_{avg}\Delta t = [(2/5)mR^2](\omega_1)/R$$
$$= [(2/5)mR^2](v_1/R)/R$$
$$= (2/5)mv_1.$$

**Note:** In this case, we had to unembed the negative sign associated with the negative *angular acceleration* because the final *angular velocity* was, itself, negative (versus the situation we had above where the two velocity terms were both positive but their difference created the negative).

Adding the two impulse equations (remember, $F\Delta t$ is impulse), we get:

$$-f_{avg}\Delta t = m(v_1 - v_o)$$

added to

$$f_{avg}\Delta t = (2/5)mv_1$$

yields

$$0 = (2/5)mv_1 + mv_1 - mv_o$$
$$= (7/5)v_1 - v_o$$
$$\Rightarrow \quad v_1 = (5/7)v_o.$$

**3.)** How can you identify a problem like this? Whenever you have translation and/or rotation coupled with slippage, you know that: $v \neq R\omega$; the frictional force acting to pull the system out of slippage will always be non-conservative; the frictional force may be external (in this case, it was); and because the conservation theorems do not hold in such situations, it's back to the basics (i.e., N.S.L.).

## K.) Parting Shot and a Bit of Order:

**1.)** For every translational parameter, there is a rotational parameter. If you are unsure what the *rotational kinetic energy* equation is, for instance, think about the *translational kinetic energy* equation and substitute in *I's* for *m's* and $\omega$*'s* for *v's*.

**2.)** Aside from forces, there are only three or four parameters you will ever be asked to determine on, say, a semester final: accelerations (angular or translational), velocities (angular or translational), distances traveled (angular or translational), and/or time of travel.

As things stand, you have a number of approaches that can generate equations that will allow you to solve for any or all of the parameters listed above. All you have to do is acquire the ability to look at a problem, decide the appropriate approach to use, and generate the needed equations.

----------------------------------------------xxxxx---------------------------------------------------

**Note from Section E:** Is the instantaneous velocity of the contact point of a rolling object really zero? To the right is a series of snapshots of a point on a ball that is rolling with constant angular velocity. Consider what happens when *the point* approaches and comes in contact with the floor. In the *y-direction*, the point transits from moving downward to moving upward. At that transition (i.e., at the contact point), the *y-component* of the point's velocity must be zero. In the *x-direction*, the net horizontal distance traveled by the point as it approaches contact gets smaller and smaller (i.e., it's slowing down), then gets larger and larger after making contact (i.e., it's speeding up). At that transition (i.e., at the contact point), the *x-component* of the point's velocity is zero. In short, the net instantaneous velocity of the point really *is* zero when it touches the ground.

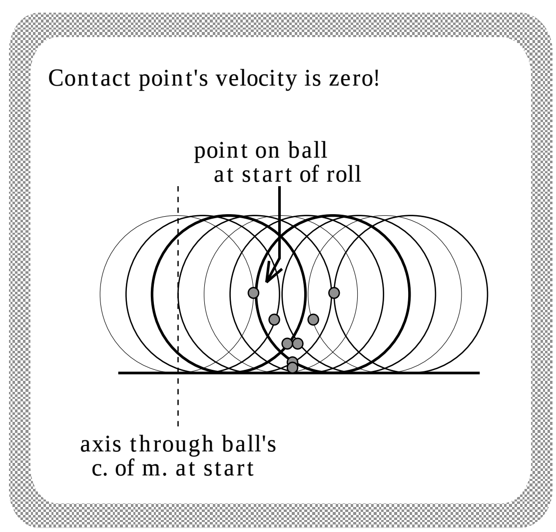


FIGURE 9.33

# QUESTIONS

**9.1)** A *frictional wheel* of mass $m = 8\ kg$, radius $R = .6$ *meters*, and moment of inertia $(1/2)mR^2$ (i.e., that of a disk) is mounted horizontally on a fixed, massless axle. The wheel is initially at rest. A rope is wrapped around the wheel's circumference and a 12 newton force is applied. The axle, whose radius is .2R, provides a 7 newton *frictional force* as the wheel rubs against it (see Figure 0). If the rope-force $F$ acts for 5 seconds:

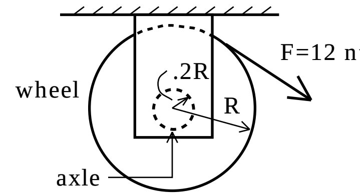

FIGURE 0

**a.)** Derive an expression for the *angular acceleration* of the wheel during that interval. Once the expression has been determined algebraically, put in the numbers.

**b.)** What is the *instantaneous translational acceleration* of a point a distance (2/3)R from the wheel's central axis during that interval?

**c.)** Using the rotational version of Newton's Second Law, derive an expression for the *angular momentum* of the wheel at the end of the period. Put in the numbers once done.

**d.)** Knowing the *angular momentum* at the end of $t = 5$ *seconds* (i.e., from *Part c*), determine the *angular velocity* of the wheel at the end of the period.

**e.)** If the *angular displacement* over the 5 second period is approximately 55.15 radians, determine the *angular velocity* at the end of the acceleration using *energy considerations.*

**f.)** Using *rotational kinematics*, show that the *angular displacement* during the acceleration interval is approximately 55.15 radians.

**g.)** Use *rotational kinematics* to verify your solution to *Part e.*

**9.2)** A block of mass $m_1 = .4$ *kilograms* sits on a frictional table (coefficient of kine-tic friction $m_k = .7$). A massless string is attached to the block, threaded over a massive pulley (mass $m_p = .08\ kg$; radius $R_p = .1875$ *meters*; and moment of inertia $I_{cm} = .5mR^2$ about the pulley's *center of mass* equal to $1.4\text{x}10^{-3}$ kg·m$^2$), and attached to a hanging mass $m_h = 1.2\ kg$ (see Figure I). If the hanging weight is allowed to freefall from rest:

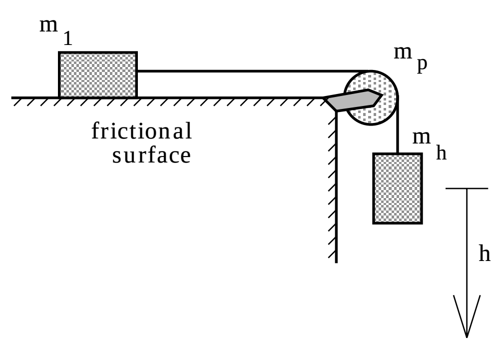

FIGURE I

**a.)** Derive an expression for the *angular acceleration* of the pulley during the freefall. Put in the numbers when you are finished.

**b.)** What is the hanging mass's *acceleration* during the freefall?

**c.)** Derive an expression for the *angular velocity* of the pulley after the hanging weight has dropped a distance equal to 1.5 meters. Do not use kinematics. Put in the numbers after you have finished the derivation.

**d.)** What is the *translational velocity* of the hanging mass after having dropped a distance equal to 1.5 meters (i.e., when the system is in the configuration outlined in *Part c*)? Don't make this hard. It isn't!

**e.)** After falling a distance of $h = 1.5$ *meters*, what is the *translational acceleration* of the hanging mass?

**f.)** Determine the *angular momentum* of the pulley after the hanging weight has fallen a distance $h = 1.5$ *meters*.

**9.3)** A beam of mass $m_b = 7\ kg$ and length $L = 1.7$ *meters* is pinned at a wall and sits at $30^o$ with the horizontal (see Figure II). A hanging mass $m_h = 3\ kg$ is attached to the beam's end, and a wire oriented at a $60^o$ angle with the horizontal is attached 2L/3 units up from the pin.

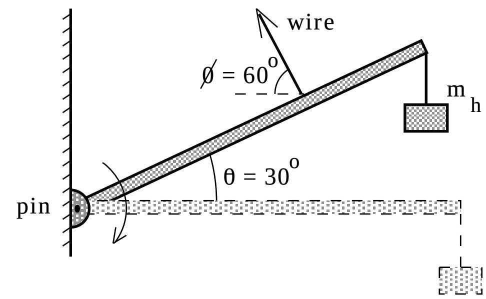


FIGURE II

**a.)** Derive expressions for the tension $T$ in the wire and the *force components* acting at the pin. Once derived, put in the numbers.

**b.)** Assuming the *moment of inertia* through the beam's *center of mass* and perpendicular to the beam's length (i.e., into the page) is equal to $(1/12)mL^2$, derive an expression for:

**i.)** The *moment of inertia* of the beam *about its pin*;

**ii.)** The *moment of inertia* of the hanging mass about the pin;

**iii.)** The *moment of inertia* of the entire system about the pin.

**c.)** The wire is cut. Derive an expression for the initial *angular acceleration* of the beam.

**d.)** Derive an expression for the *instantaneous translational acceleration* of the beam's *center of mass* just after the wire is cut.

**e.)** Derive an expression for the beam's *angular velocity* once it has reached a horizontal position. Put in the numbers once done and *do not use kinematics*.

**f.)** Determine the *translational velocity* of the *beam's center of mass* once it reaches a horizontal position.

**g.)** Determine the *angular momentum* of the system once the beam has reached the horizontal.

**h.)** Is *angular momentum* conserved? Explain.

**9.4)** A merry-go-round has a mass *m = 225 kg* and a radius *R = 2.5 meters*. Three equally spaced children push it from rest tangent to its circumference until its *angular velocity* and theirs is .8 radians/second. At that point in time, they all hop on. If we approximate the merry-go-round as a disk; if the children each have a mass equal to $m_c = 35\ kg$; and if they push with 15 newtons of force each:

**a.)** Without using kinematics, determine the *number of radians* through which the children ran during the push-period. (Hint: Think energy! Remember also that $\Delta s = R\Delta\theta$).

**b.)** Once on, the children proceed to walk from the outer-most part of the merry-go-round to a point *r = 1 meter* from the center. Determine the *angular velocity* of the merry-go-round and children once they are at *r =1 meter*.

**c.)** What quantities are conserved as the children move? Explain.

**d.)** What quantities are not conserved as the children move? Explain.

**e.)** Compare the *kinetic energy of the system* when the children were at the outer-most part of the merry-go-round and when they were at $r_1 = 1$ *meter*. Do these calculated energy values make sense in light of your response to *Parts c* and *d*? Comment.

**9.5)** The freefalling spool shown in Figure III is actually two *wheels* of radius $R_w = .04$ *meters* separated by an *axle* whose radius is $R_a = .015$ *meters*. If the mass of the system is .6 kg and the *moment of inertia* about the system's *central axis* is $I_{cm} = 1.2\text{x}10^{-4}\ kg\cdot m^2$:

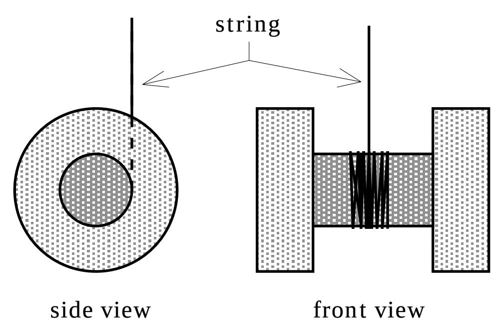

FIGURE IIIa

**a.)** Derive an expression for the *angular acceleration* of the system using $I_{cm}$ (the reason for so delineating will become evident when you read the next two parts). Put the numbers in at the end.

**b.)** Determine the *moment of inertia* of the system about an axis *parallel* to the central axis and .015 meters below it. Call this moment of inertia $I_a$.

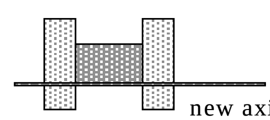

FIGURE IIIb

**c.)** Derive an expression for the *angular acceleration* of the system using $I_a$. Does this expression match the one derived in *Part a*?

**d.)** Determine the *angular velocity* of the system after the system's *center of mass* has fallen $d = .18$ *meters*. Assume the motion starts from rest, *do not* use kinematics, and note that there is a tension acting here (does this last point matter?).

**e.)** Determine the *velocity* of the *center of mass* after the system has fallen a distance $d$.

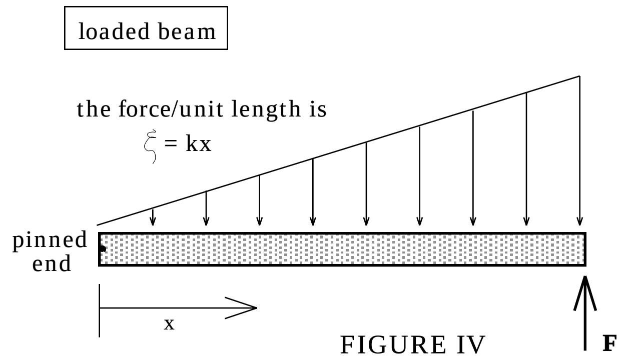

FIGURE IV

**9.6)** A loaded beam of mass $m$ and length $L$ is supported at one end by a pin and at the other end by a force $\boldsymbol{F}$ (see Figure IV). If the *force per unit length* on the beam is defined by the function $x = kx$, where $k$ is a constant with the appropriate units and $x$ is measured from the pin:

**a.)** What are $k$*'s* units?

**b.)** If $\boldsymbol{F}$ is removed, the beam will swing down. Derive an expression for the beam's angular acceleration just as it begins to move (i.e., just after $\boldsymbol{F}$ is removed).

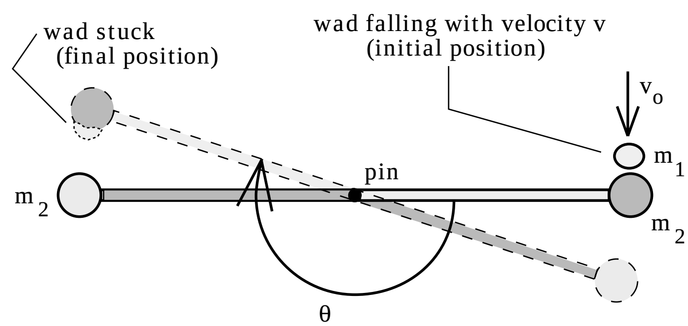

FIGURE V

**9.7)** Two bodies of mass $m_2$ each are attached to either end of an effectively massless rod of length $d$. The rod is frictionless and is pinned at its center (see Figure V to the right). A falling wad of putty whose mass is $m_1$ has velocity $v_o$ just before colliding with the far right mass as shown, sticking to that mass upon contact. If $m_1 = .9$ *kg*, $m_2 = 2$ *kg*, $d = 1.2$ *meters,* and $v_o = 2.8$ *m/s,* determine:

**a.)** The magnitude of the *angular velocity* of the rod just after the collision;

**b.)** The amount of *energy loss* that occurred during the collision, and;

**c.)** The net *angular displacement* of the system from the time just before the collision to the time when the system came to rest (assume the system does not rotate through a complete revolution).

**9.8)** A mass $m$ (take it to be a point mass) slides down a frictionless, circular incline of radius $R$ and collides with a pinned meter stick of mass $5m$ initially hanging in the vertical. After the collision, the rod rotates through an angle $\theta$ before coming to rest (see Figure VI). Assuming $R = .4d$, determine $\theta$ if:

**a.)** The mass $m$ stays at rest after the collision, and;

**b.)** The mass sticks to the rod.

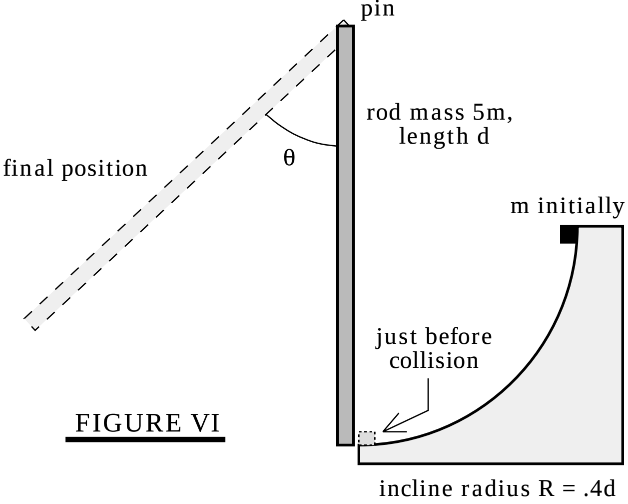


FIGURE VI

**9.9)** Disk #1 has radius $r$ and mass $m$. It initially rotates about a frictionless pin with an angular velocity of $\omega_o$. Disk #2 has radius $R$ and mass $M$. Although it is also supported by a frictionless pin, it is initially at rest. At a given instant, the pin upon which $M$ rests is moved so that the outer edge of the two disks come in contact (see Figure VII). Initially, there is slippage between the two, but finally the two come into a pure roll relative to one another (note that at that time, the velocity of the edge of each disk will be the same). In terms of $\omega_o$, determine the final angular velocity $\omega_3$ of disk #1.

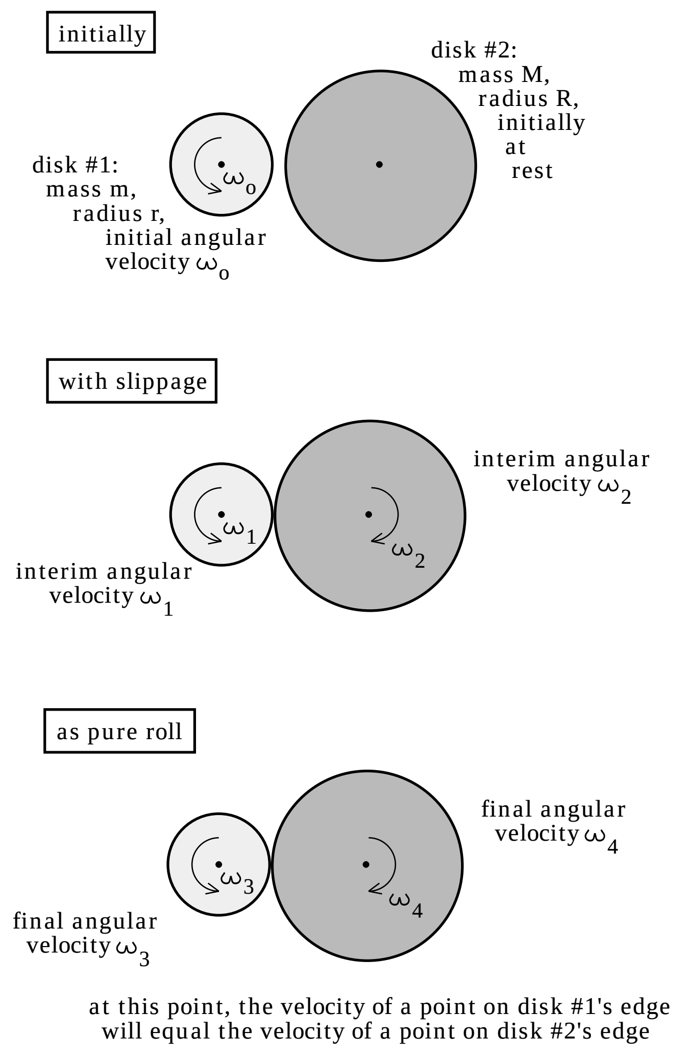


FIGURE VII